Trees and asymptotic developments for fractional stochastic differential equations

A. Neuenkirch*, I. Nourdin†, A. Rößler‡ and S. Tindel§

February 2, 2008

Abstract

In this paper we consider a n-dimensional stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H > 1/3$. After solving this equation in a rather elementary way, following the approach of [10], we show how to obtain an expansion for $E[f(X_t)]$ in terms of t, where X denotes the solution to the SDE and $f : \mathbb{R}^n \to \mathbb{R}$ is a regular function. With respect to [2], where the same kind of problem is considered, we try an improvement in three different directions: we are able to take a drift into account in the equation, we parametrize our expansion with trees (which makes it easier to use), and we obtain a sharp control of the remainder.

Keywords: fractional Brownian motion, stochastic differential equations, trees expansions.

MSC: 60H05, 60H07, 60G15

1 Introduction

In this article, we study the stochastic differential equation (SDE in short)

$$X_t^a = a + \int_0^t \sigma(X_s^a)dB_s + \int_0^t b(X_s^a)ds, \quad t \in [0,T], \tag{1}$$

*Johann Wolfgang Goethe-Universität Frankfurt am Main, FB Informatik und Mathematik, Robert-Mayer-Strasse 10, 60325 Frankfurt am Main, Germany, neuenkir@math.uni-frankfurt.de

†Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Boîte courrier 188, 4 Place Jussieu, 75252 Paris Cedex 5, France, nourdin@ccr.jussieu.fr

‡Technical University of Darmstadt, Department of Mathematics, Schlossgartenstrasse 7, 64289 Darmstadt, Germany, roessler@mathematik.tu-darmstadt.de

§Institut Élie Cartan Nancy, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France, tindel@iecn.u-nancy.fr

where B is a d-dimensional fractional Brownian motion (fBm in short) of Hurst index $H > 1/3$, $a \in \mathbb{R}^n$ is a non-random initial value and $\sigma : \mathbb{R}^n \to \mathcal{L}^{d,n}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions.

There are essentially two ways to give a sense to equation (1):

1. *Pathwise (Stratonovich) setting.* When $H > 1/2$ it is now well-known that we can use the Young integral for integration with respect to fBm and, with this choice, we have existence and uniqueness of the solution for equation (1) in the class of processes having α-Hölder continuous paths with $1 - H < \alpha < H$, see e.g. [24]. When $1/4 < H < 1/2$, it is still possible to give a sense to (1), using the rough path theory, which was initiated by Lyons [8, 9] and applied to the fBm case by Coutin and Qian [6]. In this setting, we also have existence and uniqueness in an appropriate class of processes. Remark moreover that, by using a generalization of the symmetric Russo-Vallois integral (namely the Newton-Côtes integral corrected by a Lévy area) we can obtain existence and uniqueness for (1) for any $H \in (0, 1)$, but only in dimension $n = d = 1$, see [15].

2. *Skorohod setting.* Skorohod stochastic equations, i.e., the integral with respect to fBm in (1) is understood in the Skorohod sense, are much more difficult to be solved. Indeed, until now, essentially only equations in which the noise enters linearly have been considered, see e.g., [16]. The difficulty with equations which are driven non-linearly by fBm is notorious: the Picard iteration technique involves Malliavin derivatives in such a way that the equations for estimating these derivatives cannot be closed.

In the current paper, we will solve (1) by means of a variant of the rough path theory introduced by Gubinelli in [10]. It is based on an algebraic structure, which turns out to be useful for computational purposes, but has also its own interest, and is in fact a nice alternative to the now classical theory of rough paths initiated by Lyons [8, 9]. Although SDEs of the type (1) have already been studied in [10], we include in this present paper a detailed review of the algebraic integration tools for several reasons. First of all, we want to show that this theory can simplify some aspects of the analysis of fractional equations, and we wish to give a self-contained study of these objects to illustrate this point. Moreover, the analysis of stochastic partial differential equations in [12] has lead to some clarifications with respect to [10], which may be worth presenting in the simpler finite-dimensional context. In particular, our computations will heavily rely on an Itô-type formula for the so-called weakly controlled processes, which is not included in [10], and which will be proved here in detail.

As an application of this theory of integration we study the asymptotic development with respect to t of the quantity $P_t f(a)$ defined by

$$P_t f(a) = \mathrm{E}(f(X_t^a)), \quad t \in [0,T],\, a \in \mathbb{R}^n,\, f \in \mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}), \tag{2}$$

where X^a is the solution of (1). In the case $H = 1/2$, the Taylor expansion of the semigroup P_t is well studied, see, e.g. [23, 22]. Recently, Baudoin and Coutin [2] studied the asymptotic behaviour in the case $H \neq 1/2$. In this article, we extend their result in several ways:

1. In [2], the authors considered the particular case $b \equiv 0$. Consequently, their formula contains only powers of t of the form t^{nH} with $n \in \mathbb{N}$. Due to the drift part, we obtain a more complicated expression containing powers of the type t^{nH+m} with $n, m \in \mathbb{N}$.

2. In the current article, we use rooted trees in order to obtain a nice representation of our formula. See also [23] for the case $H = 1/2$, and [11] for an application of the tree expansion to the resolution of stochastic equations.

3. In the case where $H > 1/2$, we obtain a series expansion (15) of the operator P_t, which is not only valid for small times as in [2], but for any fixed time $t \geq 0$.

Moreover, let us note that in [2], the authors used the rough paths theory of Lyons [6, 8, 9] in order to give a sense to (1). Here, as already mentioned, we use the integration theory initiated by Gubinelli [10], which allows a self-contained and hopefully a little simpler version of the essential results contained in the usual theory of integration of rough signals.

There are several reasons which motivate the study of the family of operators $(P_t,\, t \geq 0)$. For instance, the knowledge of $P_t f(a)$ for a sufficiently large class of functions f characterizes the law of the random variable X_t^a. Moreover, the knowledge of $P_t f(a)$ helps, e.g., also in finding good sample designs for the reconstruction of fractional diffusions, see [14].

The paper is organized as follows. In Section 2, we state the two main results of this paper. In section 3, the basic setup of [10] with the aim of having a self-contained introduction to the topic is recalled. In section 4, we recall some facts on the Malliavin calculus for fractional Brownian motion and some properties of stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. Finally, we give the missing proofs in section 5.

2 Main results

Before getting into a detailed description of the results contained in this article, let us first recall the main properties of a fractional Brownian motion (fBm in short). A d-dimensional fBm with Hurst parameter H is a centered Gaussian process, which can be written as

$$B = \left\{ B_t = (B_t^1, \ldots, B_t^d);\, t \geq 0 \right\}, \tag{3}$$

where $B^1, \ldots, B^d$ are d independent one-dimensional fBm, i.e., each B^i is a centered Gaussian process with continuous sample paths and covariance function

$$R_H(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H} \right) \tag{4}$$

for $i = 1, \ldots, d$. Recall also that B^i can be represented in the following way: there exists a standard Brownian motion W^i such that we have

$$B_t^i = \int_0^t K(t,s)\, dW_s^i,$$

for any $t \geq 0$, where K is the kernel given by

$$K(t,s) = c_H \left[(t-s)^{H-1/2} + \left(\frac{1}{2} - H\right) \int_s^t (u-s)^{H-3/2} \left(1 - \left(\frac{s}{u}\right)^{1/2-H}\right) du \right] \mathbf{1}_{[0,t)}(s),$$

for a constant c_H which can be expressed in terms of the Gamma function. Moreover, the fBm verifies the following two important properties:

$$\text{(scaling)} \quad \text{For any } c > 0,\ B^{(c)} = c^H B_{\cdot/c} \text{ is a fBm,} \tag{5}$$

$$\text{(stationarity)} \quad \text{For any } h > 0,\ B_{\cdot+h} - B_h \text{ is a fBm.} \tag{6}$$

2.1 Existence and uniqueness of the solution of fractional SDEs

As mentioned in the introduction, we will use for the integration with respect to fBm the integration theory developed by Gubinelli [10], on which we try to give here a simplified overview. To this purpose, will denote by $\mathcal{L}^{d,n}$ the space of linear operators from $\mathbb{R}^d$ to $\mathbb{R}^n$, i.e., the space of matrices of $\mathbb{R}^{n\times d}$.

The results for fBm we will obtain in section 4 can be summarized as follows:

Theorem 2.1. *Let B be a d-dimensional fractional Brownian motion with Hurst parameter $H > 1/3$ and $a \in \mathbb{R}^n$. Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathcal{L}^{d,n}$ be twice continuously differentiable and assume moreover that σ and b are bounded together with their derivatives. Then the stochastic differential equation*

$$X_t^a = a + \int_0^t \sigma(X_s^a)\, dB_s + \int_0^t b(X_s^a)\, ds, \qquad \textit{for} \quad t \in [0,T], \tag{7}$$

admits a unique solution in $\mathcal{Q}_{\kappa,a}(\mathbb{R}^n)$ (see Definition 3.8 below) for any $\kappa < H$ such that $2\kappa + H > 1$, where the integral $\int_0^t \sigma(X_s^a)\, dB_s$ has to be understood in the pathwise sense of Proposition 3.10. Moreover, if $f \in C^2(\mathbb{R}^n;\mathbb{R})$ is bounded together with its derivatives, then $f(X_t^a)$ can be decomposed as

$$f(X_t^a) = f(a) + \int_0^t \nabla f(X_s^a)\, b(X_s^a)\, ds + \int_0^t \nabla f(X_s^a)\, \sigma(X_s^a)\, dB_s, \tag{8}$$

for $t \in [0,T]$.

It is important to note that one of the main differences between our approach and the one developed in [6, 8] is that the latter heavily relies on the almost sure approximation of B by a sequence $\{B^n;\, n \geq 1\}$ of piecewise linear C^1-processes, while in our setting this discretization procedure is only present for the construction of the so-called fundamental map Λ (see Proposition 3.2 below).

2.2 Rooted trees and their application to the expansion of P_t

To state the next main results we need to recall some properties of stochastic rooted trees, which have been introduced in [23] in the case of standard Brownian motion.

2.2.1 Recalls on rooted trees

Definition 2.2. *A monotonically labelled S-tree (stochastic tree) $\boldsymbol{t}$ with $l = l(\boldsymbol{t}) \in \mathbb{N}$ nodes is a pair of maps $\boldsymbol{t} = (\boldsymbol{t}', \boldsymbol{t}'')$*

$$\begin{aligned} \boldsymbol{t}' &: \{2, \dots, l\} \longrightarrow \{1, \dots, l-1\} \\ \boldsymbol{t}'' &: \{1, \dots, l\} \longrightarrow \mathcal{A} \end{aligned}$$

with $\mathcal{A} = \{\gamma, \tau_0, \tau_{j_k}, k \in \mathbb{N}\}$ where j_k is a variable index with $j_k \in \{1, \dots, d\}$, such that $\boldsymbol{t}'(i) < i$, $\boldsymbol{t}''(1) = \gamma$ and $\boldsymbol{t}''(i) \in \mathcal{A} \setminus \{\gamma\}$ for $i = 2, \dots, l$. Let LTS denote the set of all monotonically labelled S-trees.

We use the following notation:

$$\begin{aligned} d(\mathbf{t}) &= |\{i : \mathbf{t}''(i) = \tau_0\}| \\ s(\mathbf{t}) &= |\{i : \mathbf{t}''(i) = \tau_{j_k}, j_k \neq 0\}| = l(\mathbf{t}) - d(\mathbf{t}) - 1 \\ \rho(\mathbf{t}) &= H\, s(\mathbf{t}) + d(\mathbf{t}) \end{aligned}$$

with $\rho(\gamma) = 0$.

In the following we denote by $LTS(S) \subset LTS$, where (S) stands for Stratonovich, the subset

$$LTS(S) = \{\mathbf{t} \in LTS : s(\mathbf{t}) = 2k, \quad k \in \mathbb{N}_0\} \tag{9}$$

with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ containing all trees having an even number of stochastic nodes.

Every monotonically labelled S-tree $\mathbf{t}$ can be represented as a graph, whose nodes are elements of $\{1, \dots, l(\mathbf{t})\}$ and whose arcs are the pairs $(\mathbf{t}'(i), i)$ for $i = 2, \dots, l(\mathbf{t})$. Here, $\mathbf{t}'$ defines a father son relation between the nodes, i.e., $\mathbf{t}'(i)$ is the father of the son i. Further, $\gamma = \otimes$ denotes the root, $\tau_0 = \bullet$ is a deterministic node and $\tau_{j_k} = \bigcirc_{j_k}$ a stochastic node. Here, we have to point out that each tree $\mathbf{t} \in LTS$ depends on the variable indices $j_1, \dots, j_{s(\mathbf{t})} \in \{1, \dots, d\}^{s(\mathbf{t})}$, although this is not mentioned explicitly if we shortly write $\mathbf{t}$ for the tree.

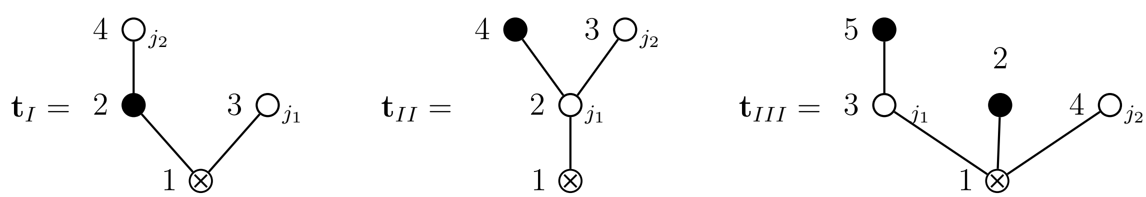

Figure 1: Some monotonically labelled trees in LTS.

Definition 2.3. *If $\boldsymbol{t}_1, \dots, \boldsymbol{t}_k$ are coloured trees then we denote by*

$$(\boldsymbol{t}_1, \dots, \boldsymbol{t}_k), \quad [\boldsymbol{t}_1, \dots, \boldsymbol{t}_k] \quad \textit{and} \quad \{\boldsymbol{t}_1, \dots, \boldsymbol{t}_k\}_j$$

the tree in which $\boldsymbol{t}_1, \dots, \boldsymbol{t}_k$ are each joined by a single branch to $\otimes$, $\bullet$ and $\bigcirc_j$, respectively (see also Figure 2).

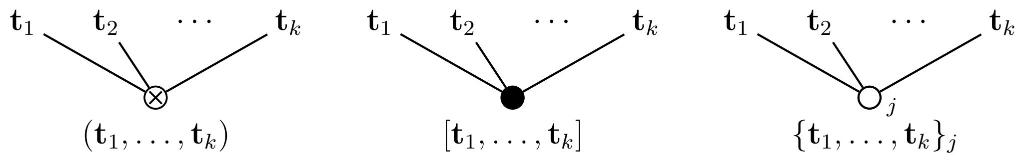

Figure 2: Writing a coloured S-tree with brackets.

Therefore proceeding recursively, for the two examples $\mathbf{t}_I$ and $\mathbf{t}_{II}$ in Figure 1 we obtain $\mathbf{t}_I = ([\bigcirc^4_{j_2}]^2, \bigcirc^3_{j_1})^1 = ([\tau^4_{j_2}]^2, \tau^3_{j_1})^1$ and $\mathbf{t}_{II} = (\{\bullet^4, \bigcirc^3_{j_2}\}^2_{j_1})^1 = (\{\tau^4_0, \tau^3_{j_2}\}^2_{j_1})^1$.

For every rooted tree $\mathbf{t} \in LTS$, there exists a corresponding elementary differential which is a direct generalization of the differential in the deterministic case, see also [23]. The elementary differential is defined recursively for some $x \in \mathbb{R}^n$ by

$$F(\gamma)(x) = f(x), \qquad F(\tau_0)(x) = b(x), \qquad F(\tau_j)(x) = \sigma^j(x),$$

for single nodes and by

$$F(\mathbf{t})(x) = \begin{cases} f^{(k)}(x) \cdot (F(\mathbf{t}_1)(x), \dots, F(\mathbf{t}_k)(x)) & \text{for } \mathbf{t} = (\mathbf{t}_1, \dots, \mathbf{t}_k) \\ b^{(k)}(x) \cdot (F(\mathbf{t}_1)(x), \dots, F(\mathbf{t}_k)(x)) & \text{for } \mathbf{t} = [\mathbf{t}_1, \dots, \mathbf{t}_k] \\ {\sigma^j}^{(k)}(x) \cdot (F(\mathbf{t}_1)(x), \dots, F(\mathbf{t}_k)(x)) & \text{for } \mathbf{t} = \{\mathbf{t}_1, \dots, \mathbf{t}_k\}_j \end{cases} \tag{10}$$

for a tree $\mathbf{t}$ with more than one node and with $\sigma^j = (\sigma^{i,j})_{1 \leq i \leq n}$ denoting the jth column of the diffusion matrix σ. Here $f^{(k)}$, $b^{(k)}$ and ${\sigma^j}^{(k)}$ define a symmetric k-linear differential operator, and one can choose the sequence of labelled S-trees $\mathbf{t}_1, \dots, \mathbf{t}_k$ in an arbitrary order. For example, the Ith component of $b^{(k)} \cdot (F(\mathbf{t}_1), \dots, F(\mathbf{t}_k))$ can be written as

$$(b^{(k)} \cdot (F(\mathbf{t}_1), \dots, F(\mathbf{t}_k)))^I = \sum_{J_1, \dots, J_k = 1}^{n} \frac{\partial^k b^I}{\partial x^{J_1} \dots \partial x^{J_k}} (F^{J_1}(\mathbf{t}_1), \dots, F^{J_k}(\mathbf{t}_k)),$$

where the components of vectors are denoted by superscript indices, which are chosen as capitals. As a result of this we get for $\mathbf{t}_I$ and $\mathbf{t}_{II}$ the elementary differentials

$$F(\mathbf{t}_I) = f''(b'(\sigma^{j_2}), \sigma^{j_1}) = \sum_{J_1, J_2 = 1}^{n} \frac{\partial^2 f}{\partial x^{J_1} \partial x^{J_2}} \left(\sum_{K_1 = 1}^{n} \frac{\partial b^{J_1}}{\partial x^{K_1}} \sigma^{K_1, j_2} \cdot \sigma^{J_2, j_1} \right),$$

$$F(\mathbf{t}_{II}) = f'({\sigma^{j_1}}''(b, \sigma^{j_2})) = \sum_{J_1 = 1}^{n} \frac{\partial f}{\partial x^{J_1}} \left(\sum_{K_1, K_2 = 1}^{n} \frac{\partial^2 \sigma^{J_1, j_1}}{\partial x^{K_1} \partial x^{K_2}} b^{K_1} \cdot \sigma^{K_2, j_2} \right).$$

Next, we assign recursively to every $\mathbf{t} \in LTS$ a multiple stochastic integral by

$$\mathcal{I}_{\mathbf{t}}(g(X^a_s))_{t_0,t} = \begin{cases} g(X^a_t) & \text{if } \mathbf{t}''(l(\mathbf{t})) = \gamma \\ \displaystyle\int_{t_0}^{t} \mathcal{I}_{\mathbf{t}-}(g(X^a_u))_{t_0,s} \, dB^j_s & \text{if } \mathbf{t}''(l(\mathbf{t})) = \tau_j \end{cases} \tag{11}$$

for $0 \leq j \leq d$ with $dB^0_s = ds$. Here, $\mathbf{t}-$ denotes the tree which is obtained from $\mathbf{t}$ by removing the last node with label $l(\mathbf{t})$.

2.2.2 Expansion of P_t with respect to time t

We will denote by $C_b^\infty(\mathbb{R}^n;\mathbb{R})$ the space of all infinitely differentiable functions $g \in C^\infty(\mathbb{R}^n;\mathbb{R})$, which are bounded together with their derivatives. Moreover set $\mathcal{A}_m = \{0,1,\dots,d\}^m$ for $m \in \mathbb{N}$, and define the differential operators $\mathcal{D}^0$ and $\mathcal{D}^j$ as

$$\mathcal{D}^0 = \sum_{k=1}^n b^k \frac{\partial}{\partial x^k} \qquad \text{and} \qquad \mathcal{D}^j = \sum_{k=1}^n \sigma^{k,j} \frac{\partial}{\partial x^k} \tag{12}$$

for $j = 1,\dots,d$. Finally, set $\mathcal{D}^\alpha = \mathcal{D}^{\alpha_1}\dots\mathcal{D}^{\alpha_m}$ for a multi-index $\alpha \in \mathcal{A}_m$.

Recall that the family of operators $(P_t,\, t \in [0,T])$ has been defined by (2). To give the expansion of P_t we will use the following assumptions on the function $f : \mathbb{R}^n \to \mathbb{R}$, the drift vector $b = (b^i)_{i=1,\dots,n}$ and the diffusion matrix $\sigma = (\sigma^{i,j})_{i=1,\dots,n,\, j=1,\dots d}$:

(A) We have $f, b^i, \sigma^{i,j} \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ for $i = 1,\dots,n$, $j = 1,\dots,d$.

The following theorem gives the expression of the expansion of P_t with respect to t:

Theorem 2.4. *1. If $H > 1/3$ and assumption (A) is satisfied, we have that*

$$\mathrm{P}_t f(a) = \sum_{\substack{\boldsymbol{t} \in LTS(S) \\ l(\boldsymbol{t}) \leq m+1}} \sum_{j_1,\dots,j_{s(\boldsymbol{t})}=1}^{d} F(\boldsymbol{t})(a)\ \mathrm{E}(\mathcal{I}_{\boldsymbol{t}}(1)_{0,1})\ t^{\rho(\boldsymbol{t})} + O(t^{(m+1)H}), \ \text{as } t \to 0 \tag{13}$$

for any $m \in \mathbb{N}_0$.

2. Let $H > 1/2$ and assumption (A) be satisfied. Moreover, assume that there exist $M \in \mathbb{N}$ and constants $K > 0$, $\kappa \in [0,1/2)$ such that

$$\sup_{\alpha \in \mathcal{A}_m} \sup_{x \in \mathbb{R}^n} |\mathcal{D}^\alpha f(x)| + \max_{i=1,\dots,n} \sup_{\alpha \in \mathcal{A}_m} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial}{\partial x_i} \mathcal{D}^\alpha f(x) \right| \leq K^m (m!)^\kappa \tag{14}$$

for all $m \geq M$. Then we have

$$\mathrm{P}_t f(a) = \sum_{\boldsymbol{t} \in LTS(S)} \sum_{j_1,\dots,j_{s(\boldsymbol{t})}=1}^{d} F(\boldsymbol{t})(a)\ \mathrm{E}(\mathcal{I}_{\boldsymbol{t}}(1)_{0,1})\ t^{\rho(\boldsymbol{t})}, \quad t \in [0,T]. \tag{15}$$

Remark 2.5. **(1)** Here, note that each tree $\mathbf{t} \in LTS(S)$ comprehends the variable indices $j_1,\dots,j_{s(\mathbf{t})}$ which can take the values $1,\dots,d$ although these variables are not mentioned explicitly by writing shortly $\mathbf{t}$ for the whole tree. The variables $j_1,\dots,j_{s(\mathbf{t})}$ correspond to the components of the driving fractional Brownian motion and appear in the second sum in the formulas (13) and (15) as well as in each tree $\mathbf{t}$ of the summands.

(2) In the case where $H > 1/2$, the boundedness of the coefficients is not needed for existence and uniqueness of the solution, see [18].

(3) Although the additional assumption (14) seems to be quite restrictive, it is however natural in a certain sense. Indeed, consider the trivial one-dimensional equation

$$dX_t^a = dB_t, \quad t \in [0,T], \qquad X_0 = a$$

for $H > 1/2$. Then we have clearly

$$X_t^a = a + B_t, \qquad t \in [0,T].$$

By the first part of Theorem 2.4 and Proposition 5.4 we have for this equation the expansion

$$\mathrm{E} f(X_t^a) = f(a) + \sum_{k=1}^{m} c_k f^{(k)}(a) t^{Hk} + O(t^{H(m+1)})$$

with

$$c_k = \frac{\mathrm{E}(B_1)^k}{k!} = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \frac{1}{2^{k/2}(k/2)!} & \text{else}. \end{cases}$$

Hence the series

$$\sum_{k=1}^{\infty} c_k f^{(k)}(a) t^{Hk}$$

converges absolutely for any $t \in [0,T]$, for instance if there exists constants $K > 0$ and $\kappa \in [0, 1/2)$ such that

$$|f^{(m)}(a)| \leq K^m (m!)^\kappa$$

for all $m \in \mathbb{N}$. But if, e.g., we have that

$$\liminf_{m \to \infty} \frac{|f^{(m)}(a)|}{(m!)^{1/2+\varepsilon}} > 0$$

with $\varepsilon > 0$, then we have

$$\sum_{k=1}^{\infty} c_k |f^{(k)}(a)| t^{Hk} = +\infty$$

for any $t \in (0,T]$. Thus the condition (14) we require for the control of the remainder is quite natural, since the coefficients of the expansion have to satisfy a similar condition, as illustrated in this example. Furthermore similar growth conditions on the remainder or the coefficients are also usual in the case $H = 1/2$, i.e., for the asymptotic expansion of Itô stochastic differential equations respectively their functionals. Compare, e.g., [3] and chapter 5 in [7].

(4) In order to solve equation (7) and to bound the Malliavin derivative of the solution in the case $H > 1/2$, we need only a boundedness condition on the first two derivatives of b and σ. To avoid too many technicalities, we have assumed in (14) that all derivatives $\mathcal{D}^\alpha f$ are uniformly bounded in $x \in \mathbb{R}^n$. However, thanks to Proposition 4.1 part (b), this condition could be relaxed, and we could allow a bound of the form

$$\sup_{\alpha \in \mathcal{A}_m} |\mathcal{D}^\alpha f(x)| + \max_{i=1,\ldots,n} \sup_{\alpha \in \mathcal{A}_m} \left| \frac{\partial}{\partial x_i} \mathcal{D}^\alpha f(x) \right| \leq K^m (m!)^\kappa \left(1 + \|x\|^q\right),$$

for a given $q \geq 0$.

For a better understanding of the previous results and as an example we consider SDE (1) in the case of $n \geq 1$, $d \geq 1$ and give an expansion of $\mathrm{P}_t f(a)$ for $m = 2$. Here, we have to consider the trees with $l(\mathbf{t}) \leq 3$ which are $\mathbf{t}_1 = \gamma^1$, $\mathbf{t}_2 = (\sigma^2_{j_1})^1$, $\mathbf{t}_3 = (\tau^2)^1$, $\mathbf{t}_4 = (\sigma^2_{j_1}, \sigma^3_{j_2})^1$, $\mathbf{t}_5 = (\{\sigma^3_{j_2}\}^2_{j_1})^1$, $\mathbf{t}_6 = ([\sigma^3_{j_1}]^2)^1$, $\mathbf{t}_7 = (\{\tau^3\}^2_{j_1})^1$, $\mathbf{t}_8 = (\tau^2, \sigma^3_{j_1})^1$, $\mathbf{t}_9 = (\tau^3, \sigma^2_{j_1})^1$, $\mathbf{t}_{10} = (\tau^2, \tau^3)^1$ and $\mathbf{t}_{11} = ([\tau^3]^2)^1$. However, only trees in $LTS(S)$ with an even number of stochastic nodes have to be included since we have $\mathrm{E}(\mathcal{I}_{\mathbf{t}}(1)_{0,1}) = 0$ for $\mathbf{t} \in LTS \setminus LTS(S)$. Then, we obtain

$$\begin{aligned}
\mathrm{P}_t f(a) = {} & F(\mathbf{t}_1)(a) + F(\mathbf{t}_3)(a)\,\mathrm{E}(\mathcal{I}_{\mathbf{t}_3}(1)_{0,1})\,t + \sum_{j_1,j_2=1}^{d} F(\mathbf{t}_4)(a)\,\mathrm{E}(\mathcal{I}_{\mathbf{t}_4}(1)_{0,1})\,t^{2H} \\
& + \sum_{j_1,j_2=1}^{d} F(\mathbf{t}_5)(a)\,\mathrm{E}(\mathcal{I}_{\mathbf{t}_5}(1)_{0,1})\,t^{2H} + F(\mathbf{t}_{10})(a)\,\mathrm{E}(\mathcal{I}_{\mathbf{t}_{10}}(1)_{0,1})\,t^2 \\
& + F(\mathbf{t}_{11})(a)\,\mathrm{E}(\mathcal{I}_{\mathbf{t}_{11}}(1)_{0,1})\,t^2 + O(t^{3H})\,.
\end{aligned}$$

Applying now (10) and (11) yields

$$\begin{aligned}
\mathrm{P}_t f(a) = {} & f(a) + f'(b)(a)\,\mathrm{E}(\int_0^1 ds)\,t + \sum_{j_1,j_2=1}^{d} f''(\sigma^{j_1}, \sigma^{j_2})(a)\,\mathrm{E}(\int_0^1 \int_0^s dB^{j_1}_{s_1}\,dB^{j_2}_s)\,t^{2H} \\
& + \sum_{j_1,j_2=1}^{d} f'(\sigma^{j_1\prime}(\sigma^{j_2}))(a)\,\mathrm{E}(\int_0^1 \int_0^s dB^{j_1}_{s_1}\,dB^{j_2}_s)\,t^{2H} + f''(b,b)(a)\,\mathrm{E}(\int_0^1 \int_0^s ds_1\,ds)\,t^2 \\
& + f'(b'(b))(a)\,\mathrm{E}(\int_0^1 \int_0^s ds_1\,ds)\,t^2 + O(t^{3H})\,,
\end{aligned}$$

which finally results in

$$\begin{aligned}
\mathrm{P}_t f(a) = {} & f(a) + \sum_{J_1=1}^{n} \frac{\partial f}{\partial x^{J_1}}(a)\,b^{J_1}(a)\,t + \frac{1}{2}\sum_{j=1}^{d} \sum_{J_1,J_2=1}^{n} \frac{\partial^2 f}{\partial x^{J_1} \partial x^{J_2}}(a)\,\sigma^{J_1,j}(a)\,\sigma^{J_2,j}(a)\,t^{2H} \\
& + \frac{1}{2}\sum_{j=1}^{d} \sum_{J_1,J_2=1}^{n} \frac{\partial f}{\partial x^{J_1}}(a)\,\frac{\partial \sigma^{J_1,j}}{\partial x^{J_2}}(a)\,\sigma^{J_2,j}(a)\,t^{2H} \\
& + \frac{1}{2}\sum_{J_1,J_2=1}^{n} \frac{\partial^2 f}{\partial x^{J_1} \partial x^{J_2}}(a)\,b^{J_1}(a)\,b^{J_2}(a)\,t^2 \\
& + \frac{1}{2}\sum_{J_1,J_2=1}^{n} \frac{\partial f}{\partial x^{J_1}}(a)\,\frac{\partial b^{J_1}}{\partial x^{J_2}}(a)\,b^{J_2}(a)\,t^2 + O(t^{3H})\,.
\end{aligned}$$

3 Some elements of algebraic integration

As already mentioned in the introduction, we include in this present section a detailed review of the algebraic integration tools contained mostly in [10, 12]. Moreover, we will give an Itô's type formula for so-called weakly controlled processes.

3.1 Increments

The extended pathwise integration we will deal with is based on the notion of 'increments', together with an elementary operator δ acting on them. The algebraic structure they generate is described in [10, 12], but here we present directly the definitions of interest for us, for sake of conciseness. First of all, for an arbitrary real number $T > 0$, a vector space V and an integer $k \geq 1$ we denote by $\mathcal{C}_k(V)$ the set of functions $g : [0,T]^k \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*(V) = \cup_{k\geq 1}\mathcal{C}_k(V)$. We can now define the announced elementary operator δ on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{16}$$

where $\hat{t}_i$ means that this particular argument is omitted. A fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Ker}\delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Im}\delta$.

Some simple examples of actions of δ, which will be the ones we will really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $s, u, t \in [0,T]$, we have

$$(\delta g)_{st} = g_t - g_s, \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}. \tag{17}$$

Furthermore, it is easily checked that $\mathcal{ZC}_{k+1}(V) = \mathcal{BC}_k(V)$ for any $k \geq 1$. In particular, the following basic property holds:

Lemma 3.1. *Let $k \geq 1$ and $h \in \mathcal{ZC}_{k+1}(V)$. Then there exists a (non unique) $f \in \mathcal{C}_k(V)$ such that $h = \delta f$.*

Proof. This elementary proof is included in [10], and will be omitted here. Let us just mention that $f_{t_1 \ldots t_k} = h_{0 t_1 \ldots t_k}$ is a possible choice. □

Observe that Lemma 3.1 implies that all the elements $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non unique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_2(V)}$: it measures how much a given 1-increment is far from being an exact increment of a function, i.e., a finite difference.

Notice that our future discussions will mainly rely on k-increments with $k \leq 2$, for which we will use some analytical assumptions. Namely, we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_\mu = \sup_{s,t\in[0,T]} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V);\ \|f\|_\mu < \infty\}.$$

Obviously, the usual Hölder spaces $\mathcal{C}_1^\mu(V)$ will be determined in the following way: for a continuous function $g \in \mathcal{C}_1(V)$, we simply set

$$\|g\|_\mu = |\delta g|_\mu, \tag{18}$$

and we will say that $g \in \mathcal{C}_1^\mu(V)$ iff $\|g\|_\mu$ is finite. Notice that $\|\cdot\|_\mu$ is only a semi-norm on $\mathcal{C}_1(V)$, but we will generally work on spaces of the type

$$\mathcal{C}_{1,a}^\mu(V) = \{g : [0,T] \to V;\, g_0 = a,\, \|g\|_\mu < \infty\}\,, \tag{19}$$

for a given $a \in V$, on which $\|g\|_\mu$ thus becomes a norm. For $h \in \mathcal{C}_3(V)$ set in the same way

$$\begin{aligned} \|h\|_{\gamma,\rho} &= \sup_{s,u,t\in[0,T]} \frac{|h_{sut}|}{|u-s|^\gamma |t-u|^\rho} \\ \|h\|_\mu &= \inf\left\{ \sum_i \|h_i\|_{\rho_i,\mu-\rho_i};\, h = \sum_i h_i,\, 0<\rho_i<\mu \right\}, \end{aligned} \tag{20}$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0,z)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V);\, \|h\|_\mu < \infty\}\,.$$

Eventually, let $\mathcal{C}_3^{1+}(V) = \cup_{\mu>1}\mathcal{C}_3^\mu(V)$, and remark that the same kind of norms can be considered on the spaces $\mathcal{Z}\mathcal{C}_3(V)$, leading to the definition of some spaces $\mathcal{Z}\mathcal{C}_3^\mu(V)$ and $\mathcal{Z}\mathcal{C}_3^{1+}(V)$.

With these notations in mind the following proposition is a basic result, which belongs to the core of our approach to pathwise integration. Its proof may be found in a simple form in [12].

Proposition 3.2 (The Λ-map)**.** *There exists a unique linear map $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+}(V) \to \mathcal{C}_2^{1+}(V)$ such that*

$$\delta\Lambda = Id_{\mathcal{Z}\mathcal{C}_3^{1+}(V)} \qquad \text{and} \qquad \Lambda\delta = Id_{\mathcal{C}_2^{1+}(V)}.$$

In other words, for any $h \in \mathcal{C}_3^{1+}(V)$ such that $\delta h = 0$ there exists a unique $g = \Lambda(h) \in \mathcal{C}_2^{1+}(V)$ such that $\delta g = h$. Furthermore, for any $\mu > 1$, the map Λ is continuous from $\mathcal{Z}\mathcal{C}_3^\mu(V)$ to $\mathcal{C}_2^\mu(V)$ and we have

$$\|\Lambda h\|_\mu \le \frac{1}{2^\mu - 2}\|h\|_\mu, \qquad h \in \mathcal{Z}\mathcal{C}_3^\mu(V). \tag{21}$$

We can now give an algorithm for a canonical decomposition of a function $g \in \mathcal{C}_2(V)$, whose increment δg is smooth enough:

Corollary 3.3. *Let $g \in \mathcal{C}_2(V)$ such that $\delta g \in \mathcal{C}_3^\mu(V)$ for $\mu > 1$. Then, for an arbitrary $a \in V$, g can be decomposed in a unique way as*

$$g = \delta f + \Lambda\delta g, \tag{22}$$

where $f \in \mathcal{C}_{1,a}(V)$.

Proof. This proof is elementary. We include it here in order to see some simple manipulations of the objects we have introduced so far.

The existence of the decomposition is due to the following fact: if $\delta g \in \mathcal{C}_3^\mu(V)$, then it belongs to $\mathrm{Dom}(\Lambda)$. Thus, let us set $h = \Lambda\delta g$. Then $\delta(g-h) = 0$, which means that $g - h \in \mathcal{ZC}_2$, and since $\mathcal{ZC}_2 = \mathcal{BC}_1$, there exists an element $f \in \mathcal{C}_1$ such that $g - h = \delta f$. Hence we have obtained a decomposition of the form (22).

As far as the uniqueness of the decomposition is concerned, if f^1, f^2 satisfy (22), then $\delta f^1 = \delta f^2$ and hence they differ only by a constant. Since f^1, f^2 are both supposed to be elements of $\mathcal{C}_{1,a}(V)$, where a is a fixed initial condition, we obtain $f^1 = f^2$, which proves our claim.

□

Let us mention at this point a first link between the structures we have introduced so far and the problem of integration of irregular functions.

Corollary 3.4. *For any 1-increment $g \in \mathcal{C}_2(V)$ such that $\delta g \in \mathcal{C}_3^{1+}$, set $\delta f = (Id - \Lambda\delta)g$. Then*

$$(\delta f)_{st} = \lim_{|\Pi_{ts}| \to 0} \sum_{i=0}^{n} g_{t_i\, t_{i+1}},$$

where the limit is over any partition $\Pi_{ts} = \{t_0 = t, \dots, t_n = s\}$ of $[t,s]$, whose mesh tends to zero. Thus, the 1-increment δf is the indefinite integral of the 1-increment g.

Proof. Just consider the equation $g = \delta f + \Lambda\delta g$ and write

$$\begin{aligned} S_\Pi = \sum_{i=0}^{n} g_{t_i\, t_{i+1}} &= \sum_{i=0}^{n} (\delta f)_{t_i\, t_{i+1}} + \sum_{i=0}^{n} (\Lambda\delta g)_{t_i\, t_{i+1}} \\ &= (\delta f)_{st} + \sum_{i=0}^{n} (\Lambda\delta g)_{t_i\, t_{i+1}}. \end{aligned}$$

Then observe that, due to the fact that $\Lambda\delta g \in \mathcal{C}_3^{1+}(V)$, the last sum converges to zero. □

3.2 Computations in $\mathcal{C}_*$

Let us specialize now to the case $V = \mathbb{R}^d$ for $d \geq 1$. We will also denote by $\mathcal{L}^{d,l}$ the space of linear operators from $\mathbb{R}^d$ to $\mathbb{R}^l$, i.e., the space of matrices of $\mathbb{R}^{l\times d}$ and set $\mathcal{C}_k\mathcal{L}^{d,l} = \mathcal{C}_k(\mathcal{L}^{d,l})$. Then $(\mathcal{C}_*, \delta)$ can be endowed with the following product: for $g \in \mathcal{C}_n\mathcal{L}^{d,l}$ and $h \in \mathcal{C}_m(\mathbb{R}^d)$ let gh be the element of $\mathcal{C}_{n+m-1}(\mathbb{R}^l)$ defined by

$$(gh)_{t_1,\dots,t_{m+n+1}} = g_{t_1,\dots,t_n} h_{t_n,\dots,t_{m+n-1}}, \quad t_1, \dots, t_{m+n-1} \in [0,T]. \tag{23}$$

In this context, we have the following useful properties.

Proposition 3.5. *The following differentiation rules hold true:*

1. Let $g \in \mathcal{C}_1\mathcal{L}^{d,l}$ and $h \in \mathcal{C}_1(\mathbb{R}^d)$. Then $gh \in \mathcal{C}_1(\mathbb{R}^l)$ and

$$\delta(gh) = \delta g\, h + g\, \delta h. \tag{24}$$

2. Let $g \in \mathcal{C}_1\mathcal{L}^{d,l}$ and $h \in \mathcal{C}_2(\mathbb{R}^d)$. Then $gh \in \mathcal{C}_2(\mathbb{R}^l)$ and

$$\delta(gh) = \delta g\, h - g\, \delta h. \tag{25}$$

3. Let $g \in \mathcal{C}_2\mathcal{L}^{d,l}$ and $h \in \mathcal{C}_1(\mathbb{R}^d)$. Then $gh \in \mathcal{C}_2(\mathbb{R}^l)$ and

$$\delta(gh) = \delta g\, h + g\, \delta h. \tag{26}$$

Proof. We will just prove (24), the other relations being just as simple. If $g, h \in \mathcal{C}_1$, then

$$[\delta(gh)]_{st} = g_t h_t - g_s h_s = g_s\,(h_t - h_s) + (g_t - g_s)\, h_t = g_s\,(\delta h)_{st} + (\delta g)_{st}\, h_t,$$

which proves our claim. □

The iterated integrals of smooth functions on $[0,T]$ are obviously particular cases of elements of $\mathcal{C}$, which will be of interest for us. Let us recall some basic rules for these objects: consider $f \in \mathcal{C}_1^\infty\mathcal{L}^{d,l}$ and $g \in \mathcal{C}_1^\infty(\mathbb{R}^d)$, where $\mathcal{C}_1^\infty$ denotes the set of smooth functions on $[0,T]$. Then the integral $\int f\, dg$, which will be denoted by $\mathcal{J}(f\, dg)$, can be considered as an element of $\mathcal{C}_2^\infty(\mathbb{R}^l)$. Namely, for $s,t \in [0,T]$ we set

$$\mathcal{J}_{st}(f\, dg) = \left(\int f dg\right)_{st} = \int_s^t f_u dg_u.$$

The multiple integrals can also be defined in the following way: given a smooth element $h \in \mathcal{C}_2^\infty\mathcal{L}^{d,l}$ and $s,t \in [0,T]$, we set

$$\mathcal{J}_{st}(h\, dg) \equiv \left(\int h dg\right)_{st} = \int_s^t h_{su} dg_u.$$

In particular, for $f^1 \in \mathcal{C}_1^\infty(\mathbb{R}^{d_1})$, $f^2 \in \mathcal{C}_1^\infty\mathcal{L}^{d_1,d_2}$ and $f^3 \in \mathcal{C}_1^\infty\mathcal{L}^{d_2,d_3}$ the double integral $\mathcal{J}_{st}(f^3\, df^2 df^1)$ is defined as

$$\mathcal{J}_{st}(f^3\, df^2 df^1) = \left(\int f^3\, df^2 df^1\right)_{st} = \int_s^t \mathcal{J}_{su}\left(f^3\, df^2\right)\, df_u^1.$$

Now suppose that the nth order iterated integral of $f^{n+1} df^n \cdots df^2$, which is denoted by $\mathcal{J}(f^{n+1} df^n \cdots df^2)$, has been defined for $f^j \in \mathcal{C}_1^\infty\mathcal{L}^{d_{j-1},d_j}$. Then, if $f^1 \in \mathcal{C}_1^\infty(\mathbb{R}^{d_1})$, we set

$$\mathcal{J}_{st}(f^{n+1} df^n \cdots df^2 df^1) = \int_s^t \mathcal{J}_{su}\left(f^{n+1} df^n \cdots df^2\right)\, df_u^1, \tag{27}$$

which recursively defines the iterated integrals of smooth functions. Observe that a nth order integral $\mathcal{J}(df^n \cdots df^2 df^1)$ could be defined along the same lines.

The following relations between multiple integrals and the operator δ will also be useful:

Proposition 3.6. *Let $f \in \mathcal{C}_1^\infty \mathcal{L}^{d,l}$ and $g \in \mathcal{C}_1^\infty(\mathbb{R}^d)$. Then it holds that*

$$\delta g = \mathcal{J}(dg), \qquad \delta\left(\mathcal{J}(f dg)\right) = 0, \qquad \delta\left(\mathcal{J}(df dg)\right) = -(\delta f)(\delta g) = -\mathcal{J}(df)\mathcal{J}(dg),$$

and

$$\delta\left(\mathcal{J}(df^n \cdots df^1)\right) = -\sum_{i=1}^{n-1} \mathcal{J}\left(df^n \cdots df^{i+1}\right) \mathcal{J}\left(df^i \cdots df^1\right).$$

Proof. Here the proof is elementary again. We will just show the third of the relations. For $s,t \in [0,T]$ we have

$$\mathcal{J}_{st}(dgdf) = \int_s^t (f_u - f_s) dg_u = \int_s^t f_u dg_u - K_{st},$$

with $K_{st} = f_s(g_t - g_s)$. The first term of the right hand side is easily seen to be in $\mathcal{ZC}_2$. Thus

$$\delta\left(\mathcal{J}(dgdf)\right)_{sut} = -\left(\delta K\right)_{sut} = -[f_u - f_s][g_t - g_u],$$

which gives the announced result. □

3.3 Weakly controlled processes

Recall that we have in mind to solve equations of the form

$$dy_t = \sigma(y_t) dx_t, \qquad y_0 = a, \tag{28}$$

where $t \in [0,T]$, y is a $\mathbb{R}^l$-valued continuous process, $\sigma : \mathbb{R}^l \to \mathcal{L}^{d,l}$ is a C_b^2 function, i.e. twice continuously differentiable and bounded together with its derivatives, x is a $\mathbb{R}^d$-valued path and $a \in \mathbb{R}^l$ is a fixed initial condition. As usual in rough path type considerations, we will have to assume a priori the following hypothesis in order to handle equations like (28):

Hypothesis 3.7. *The path x is $\mathbb{R}^d$-valued γ-Hölder with $\gamma > 1/3$ and admits a Lévy area, that is a process $\mathbf{x^2} = \mathcal{J}(dxdx) \in \mathcal{C}_2^{2\gamma}\mathcal{L}^{d,d}$ satisfying*

$$\delta \mathbf{x^2} = \delta x \otimes \delta x, \quad \textit{i.e.} \quad \left[(\delta \mathbf{x^2})_{sut}\right](i,j) = [\delta x^i]_{su}[\delta x^j]_{ut}, \quad s,u,t \in [0,T],\ i,j \in \{1,\ldots,d\}.$$

The solution to (28) will then be expressed as a continuous function of the input a, σ, x and $\mathbf{x^2}$.

Let us now be more specific about the global strategy, we will use to solve equation (28). First of all, simple heuristic considerations show that, if the equation admits a solution, it should be a weakly controlled path, i.e., a process of the following form:

Definition 3.8. *Let z be a process in $\mathcal{C}_1^\kappa(\mathbb{R}^k)$ with $\kappa \le \gamma$ and $2\kappa + \gamma > 1$. We say that z is a weakly controlled path based on x, if $z_0 = a$, which is a given initial condition in $\mathbb{R}^k$, and $\delta z \in \mathcal{C}_2^\kappa(\mathbb{R}^k)$ can be decomposed into*

$$\delta z = \zeta \delta x + r, \quad \textit{i.e.} \quad (\delta z)_{st} = \zeta_s (\delta x)_{st} + r_{st}, \quad s,t \in [0,T], \tag{29}$$

with $\zeta \in \mathcal{C}_1^\kappa \mathcal{L}^{d,k}$ and r is a regular part such that $r \in \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)$. The space of weakly controlled paths will be denoted by $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$, and a process $z \in \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ can be considered in fact as a couple (z,ζ). The natural semi-norm on $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ is given by

$$\mathcal{N}[z; \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)] = \mathcal{N}[z; \mathcal{C}_1^\kappa(\mathbb{R}^k)] + \mathcal{N}[\zeta; \mathcal{C}_1^\infty \mathcal{L}^{d,k}] + \mathcal{N}[\zeta; \mathcal{C}_1^\kappa \mathcal{L}^{d,k}] + \mathcal{N}[r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)]$$

with $\mathcal{N}[g; \mathcal{C}_1^\kappa]$ defined by (18) and $\mathcal{N}[\zeta; \mathcal{C}_1^\infty(V)] = \sup_{0\le s\le T} |\zeta_s|_V$.

Note that it is always possible to find $\kappa \le \gamma$ with $2\kappa + \gamma > 1$, since $\gamma > 1/3$. With this definition at hand, we will try to solve equation (28) in the following way:

1. Study the stability of $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ under a smooth map $\varphi : \mathbb{R}^k \to \mathbb{R}^n$.
2. Define rigorously the integral $\int z_u dx_u = \mathcal{J}(z dx)$ for a weakly controlled path z and compute its decomposition (29).
3. Solve equation (28) in the space $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ by a fixed point argument.

In this section, we will concentrate on the first two points of this program.

Let us first see, how smooth functions act on weakly controlled paths:

Proposition 3.9. *Let $z \in \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ with decomposition (29), $\varphi \in C_b^2(\mathbb{R}^k; \mathbb{R}^n)$ and set $\hat z = \varphi(z)$, $\hat a = \varphi(a)$. Then $\hat z \in \mathcal{Q}_{\kappa,\hat a}(\mathbb{R}^n)$, and it can be decomposed into*

$$\delta \hat z = \hat\zeta \delta x + \hat r,$$

with

$$\hat\zeta = \nabla\varphi(z)\zeta \quad \text{and} \quad \hat r = \nabla\varphi(z) r + [\delta(\varphi(z)) - \nabla\varphi(z)\delta z].$$

Furthermore,

$$\mathcal{N}[\hat z; \mathcal{Q}_{\kappa,\hat a}(\mathbb{R}^n)] \le c_{\varphi,T}\left(1 + \mathcal{N}^2[z; \mathcal{Q}_{\kappa,a}(\mathbb{R}^n)]\right). \tag{30}$$

Proof. The algebraic part of the assertion is quite straightforward. Just write

$$\begin{aligned}(\delta\hat z)_{st} &= \varphi(z_t) - \varphi(z_s) = \nabla\varphi(z_s)(\delta z)_{st} + \varphi(z_t) - \varphi(z_s) - \nabla\varphi(z_s)(\delta z)_{st}\\ &= \nabla\varphi(z_s)\zeta_s(\delta x)_{st} + \nabla\varphi(z_s) r_{st} + \varphi(z_t) - \varphi(z_s) - \nabla\varphi(z_s)(\delta z)_{st}\\ &= \hat\zeta_s(\delta x)_{st} + \hat r_{st},\end{aligned}$$

which is the desired decomposition.

In order to give an estimate for $\mathcal{N}[\hat z; \mathcal{Q}_{\kappa,\hat a}(\mathbb{R}^n)]$, one has of course to establish bounds for $\mathcal{N}[\hat z; \mathcal{C}_1^\kappa(\mathbb{R}^n)]$, $\mathcal{N}[\hat\zeta; \mathcal{C}_1^\kappa \mathcal{L}^{d,n}]$, $\mathcal{N}[\hat\zeta; \mathcal{C}_1^\infty \mathcal{L}^{d,n}]$ and $\mathcal{N}[\hat r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^n)]$. Let us focus on the last of these estimates, the other ones are quite similar. First notice that $\hat r = \hat r^1 + \hat r^2$ with

$$\hat r^1_{st} = \nabla\varphi(z_s) r_{st} \quad \text{and} \quad \hat r^2_{st} = \varphi(z_t) - \varphi(z_s) - \nabla\varphi(z_s)(\delta z)_{st}. \tag{31}$$

Now, since $\nabla\varphi$ is a bounded $\mathcal{L}^{k,n}$-valued function, we have

$$\mathcal{N}[\hat r^1; \mathcal{C}_2^{2\kappa}(\mathbb{R}^n)] \le \|\nabla\varphi\|_\infty \mathcal{N}[r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)]. \tag{32}$$

Moreover,

$$|\hat{r}^2_{st}| \leq \frac{1}{2}\|\nabla^2\varphi\|_\infty |(\delta z)_{st}|^2 \leq c_\varphi \mathcal{N}^2[z; \mathcal{C}_1^\kappa(\mathbb{R}^k)]|t-s|^{2\kappa},$$

which yields

$$\mathcal{N}[\hat{r}^2; \mathcal{C}_2^{2\kappa}(\mathbb{R}^n)] \leq c_\varphi \mathcal{N}^2[r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)], \tag{33}$$

and thus we obtain

$$\mathcal{N}[\hat{r}; \mathcal{C}_2^{2\kappa}(\mathbb{R}^n)] \leq c_\varphi \left(1 + \mathcal{N}^2[r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)]\right),$$

which ends the proof. □

Let us now turn to the integration of weakly controlled paths, which is summarized in the following proposition. Notice that below we will use two additional notations. We will set M^* for the transposition of a matrix M and denote by $M \cdot N$ the inner product of two vectors or two matrices.

Proposition 3.10. *For a given $\gamma > 1/3$ and $\kappa < \gamma$, let x be a process satisfying Hypothesis 3.7. Furthermore, let $m \in \mathcal{Q}_{\kappa,b}(\mathcal{L}^{d,1})$ with decomposition $m_0 = b \in \mathcal{L}^{d,1}$ and*

$$(\delta m)_{st} = [\mu_s (\delta x)_{st}]^* + r_{st}, \qquad \textit{where} \qquad \mu \in \mathcal{C}_1^\kappa \mathcal{L}^{d,d},\ r \in \mathcal{C}_2^{2\kappa}\mathcal{L}^{d,1}. \tag{34}$$

Define z by $z_0 = a \in \mathbb{R}$ and

$$\delta z = m\, \delta x + \mu \cdot \mathbf{x^2} - \Lambda(r\, \delta x + \delta\mu \cdot \mathbf{x^2}). \tag{35}$$

Finally, set

$$\mathcal{J}(m\, dx) = \delta z. \tag{36}$$

Then:

1. *z is well-defined as an element of $\mathcal{Q}_{\kappa,a}(\mathbb{R})$.*
2. *The semi-norm of z in $\mathcal{Q}_{\kappa,a}(\mathbb{R})$ can be estimated as*

$$\mathcal{N}[z; \mathcal{Q}_{\kappa,a}(\mathbb{R})] \leq c_x \left(1 + T^{\gamma-\kappa}\mathcal{N}[m; \mathcal{Q}_{\kappa,b}(\mathcal{L}^{d,1})]\right), \tag{37}$$

for a positive constant c_x depending only on x and $\mathbf{x^2}$. The constant c_x can be bounded as follows:

$$c_x \leq c\left[|x|_\gamma + |\mathbf{x^2}|_{2\gamma}\right], \qquad \textit{for a universal constant } c.$$

Moreover, we have

$$\|\delta z\|_\kappa \leq c_x T^{\gamma-\kappa}\mathcal{N}[m; \mathcal{Q}_{\kappa,b}(\mathcal{L}^{d,1})]. \tag{38}$$

3. *It holds*

$$\mathcal{J}_{st}(m\, dx) = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n} \left[m_{t_i}(\delta x)_{t_i,t_{i+1}} + \mu_{t_i} \cdot \mathbf{x^2}_{t_i,t_{i+1}}\right] \tag{39}$$

for any $0 \leq s < t \leq T$, where the limit is taken over all partitions $\Pi_{st} = \{s = t_0, \dots, t_n = t\}$ of $[s,t]$, as the mesh of the partition goes to zero.

Before going into the technical details of the proof, let us see how to recover (35) in the smooth case, in order to justify our definition of the integral. (Notice however that (39) corresponds to the usual definition in the rough paths theory [8], which gives another kind of justification.)

Let us assume for the moment that x is a smooth function and that $m \in \mathcal{C}_1^\infty(\mathcal{L}^{d,1})$ admits the decomposition (34) with $\mu \in \mathcal{C}_1^\infty \mathcal{L}^{d,d}$ and $r \in \mathcal{C}_2^\infty \mathcal{L}^{d,1}$. Then $\mathcal{J}(m\, dx)$ is well-defined, and we have

$$\int_s^t m_u dx_u = m_s[x_t - x_s] + \int_s^t [m_u - m_s] dx_u$$

for $s \le t$, respectively

$$\mathcal{J}(m\, dx) = m\, \delta x + \mathcal{J}(\delta m\, dx).$$

Let us now plug the decomposition (34) into this expression, which yields

$$\begin{aligned} \mathcal{J}(m\, dx) &= m\, \delta x + \mathcal{J}((\mu \delta x)^*\, dx) + \mathcal{J}(r\, dx) \\ &= m\, \delta x + \mu \cdot \mathbf{x^2} + \mathcal{J}(r\, dx). \end{aligned} \tag{40}$$

For the sake of clarity, let us give some details about the identity $\mathcal{J}((\mu\delta x)^*\, dx) = \mu \cdot \mathbf{x^2}$. Indeed, according to our definitions in Section 3.2 we have

$$\mathcal{J}_{st}((\mu\delta x)^*\, dx) = \int_s^t ([\delta x]_{su}^* \mu_s)\, dx_u = \int_s^t \mu_s \cdot [(\delta x)_{su} \otimes dx_u] = \mu_s \cdot \int_s^t [\delta x]_{su} \otimes dx_u = \mu_s \cdot \mathbf{x}^{\mathbf{2}}_{st}$$

for $s \le t$, which proves the announced identity. Notice also that the terms $m\, \delta x$ and $\mu\, \mathbf{x^2}$ in (40) are well-defined as soon as x and $\mathbf{x^2}$ are defined themselves. In order to push forward our analysis to the rough case, it remains to handle the term $\mathcal{J}(r\, dx)$. Thanks to (40) we can write

$$\mathcal{J}(r\, dx) = \mathcal{J}(m\, dx) - m\, \delta x - \mu \cdot \mathbf{x^2},$$

and let us analyze this relation by applying δ to both sides. Using the second part of Proposition 3.5 and the whole Proposition 3.6 yields

$$\begin{aligned} \delta\left[\mathcal{J}(r\, dx)\right] &= -\delta\left[m\, \delta x\right] - \delta\left[\mu \cdot \mathbf{x^2}\right] \\ &= -\delta m\, \delta x - \delta\mu \cdot \mathbf{x^2} + \mu \cdot (\delta x \otimes \delta x) \\ &= -\left[(\mu\, \delta x)^* + r\right] \delta x - \delta\mu \cdot \mathbf{x^2} + \mu \cdot (\delta x \otimes \delta x) \\ &= -\delta\mu \cdot \mathbf{x^2} - r\, \delta x. \end{aligned} \tag{41}$$

Assuming now that $\delta\mu \cdot \mathbf{x^2}$ and $r\, \delta x$ are both elements of $\mathcal{C}_2^\mu$ with $\mu > 1$, $\delta\mu \cdot \mathbf{x^2} + r\, \delta x$ becomes an element of $\mathrm{Dom}(\Lambda)$, and inserting (41) into (40) we obtain

$$\delta z = \mathcal{J}(m\, dx) \equiv m\, \delta x + \mu \cdot \mathbf{x^2} - \Lambda(r\, \delta x + \delta\mu \cdot \mathbf{x^2}),$$

which is the expression (35) of our Proposition 3.10. Thus (35) is a natural expression for $\mathcal{J}(m\, dx)$.

Proof of Proposition 3.10. We will decompose this proof in two steps.

Step 1: Recalling the assumption $2\kappa+\gamma>1$, let us analyze the three terms in the right hand side of (35) and show that they define an element of $\mathcal{Q}_{\kappa,a}$ such that $\delta z=\zeta\delta x+r$ with

$$\zeta=m \quad \text{and} \quad \hat{r}=\mu\cdot\mathbf{x^2}-\Lambda\left(r\,\delta x+\delta\mu\cdot\mathbf{x^2}\right).$$

Indeed, on the one hand $m\in\mathcal{C}_1^\kappa\mathcal{L}^{d,1}$ and thus $\zeta=m$ is of the desired form for an element of $\mathcal{Q}_{\kappa,a}$. On the other hand, if $m\in\mathcal{Q}_{\kappa,b}$, μ is assumed to be bounded and since $\mathbf{x^2}\in\mathcal{C}_2^{2\kappa}\mathcal{L}^{d,d}$ we get that $\mu\cdot\mathbf{x^2}\in\mathcal{C}_2^{2\gamma}(\mathbb{R})$. Along the same lines we can prove that $r\,\delta x\in\mathcal{C}_3^{2\kappa+\gamma}(\mathbb{R})$ and $\delta\mu\cdot\mathbf{x^2}\in\mathcal{C}_3^{\kappa+2\gamma}(\mathbb{R})$. Since $\kappa+2\gamma\geq 2\kappa+\gamma>1$, we obtain that $r\,\delta x+\delta\mu\cdot\mathbf{x^2}\in\mathrm{Dom}(\Lambda)$ and

$$\Lambda\left(r\,\delta x+\delta\mu\,\mathbf{x^2}\right)\in\mathcal{C}_2^{2\kappa+\gamma}(\mathbb{R}).$$

Thus we have proved that

$$r=\mu\cdot\mathbf{x^2}-\Lambda\left(r\,\delta x+\delta\mu\cdot\mathbf{x^2}\right)\in\mathcal{C}_2^{2\kappa}(\mathbb{R})$$

and hence that $z\in\mathcal{Q}_{\kappa,a}(\mathbb{R})$. The estimates (37) and (38) are now obtained using to the same kind of considerations and are left to the reader for the sake of conciseness.

Step 2: The same kind of computations as those leading to (41) also show that

$$\delta\left(m\,\delta x+\mu\cdot\mathbf{x^2}\right)=-\left[r\,\delta x+\delta\mu\cdot\mathbf{x^2}\right].$$

Hence equation (35) can also be read as

$$\mathcal{J}(m\,dx)=[\mathrm{Id}-\Lambda\delta]\,(m\,\delta x+\mu\cdot\mathbf{x^2}),$$

and a direct application of Corollary 3.4 yields (39), which ends our proof. □

Notice that the previous proposition has a straightforward multidimensional extension, which we state for further use:

Corollary 3.11. *Let x be a process satisfying Hypothesis 3.7 and let $m\in\mathcal{Q}_{\kappa,b}(\mathcal{L}^{d,k})$ with decomposition $m_0=b\in\mathcal{L}^{d,k}$ and*

$$(\delta m^i)_{st}=\left[\mu_s^i(\delta x)_{st}\right]^*+r_{st}^i \quad \textit{with} \quad \mu^i\in\mathcal{C}_1^\kappa\mathcal{L}^{d,d},\ r^i\in\mathcal{C}_2^{2\kappa}\mathcal{L}^{d,k}, \quad i=1,\dots,k, \tag{42}$$

where we have considered m as a $\mathbb{R}^{k\times d}$-valued path and have set $m^i=m(i,\cdot)$. Define z by $z_0=a\in\mathbb{R}^k$ and

$$\delta z^i=\mathcal{J}(m^i\,dx)\equiv m^i\,\delta x+\mu^i\cdot\mathbf{x^2}+\Lambda(r^i\,\delta x+\delta\mu^i\cdot\mathbf{x^2}). \tag{43}$$

Then the conclusions of Proposition 3.10 still hold in this context.

Notice also that our extended pathwise integral has a nice continuity property with respect to the driving path x. See also [10, p. 14].

Proposition 3.12. *Let x be a function satisfying Hypothesis 3.7 and assume that there exists a sequence $\{x^n;\, n \ge 1\}$ of piecewise C^1-functions from $[0,T]$ to $\mathbb{R}^d$ such that*

$$\lim_{n\to\infty} \mathcal{N}[x^n - x; \mathcal{C}_1^\gamma(\mathbb{R}^d)] = 0, \qquad \textit{and} \qquad \lim_{n\to\infty} \mathcal{N}[\mathbf{x}^{\mathbf{2},n} - \mathbf{x}^{\mathbf{2}}; \mathcal{C}_2^{2\gamma}\mathcal{L}^{d,d}] = 0. \tag{44}$$

For $n \ge 1$, define $z^n \in \mathcal{C}_1^\kappa(\mathbb{R}^k)$ in the following way: set $z_0 = b \in \mathbb{R}^k$ and

$$\delta z^n = \zeta^n \delta x^n + r^n,$$

where $\zeta^n \in \mathcal{C}_1^\kappa \mathcal{L}^{d,k}$ and $r^n \in \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)$. Moreover, let z be a weakly controlled process with decomposition (29) and assume that

$$\lim_{n\to\infty} \mathcal{N}[z^n - z; \mathcal{C}_1^\kappa(\mathbb{R}^k)] + \mathcal{N}[\zeta^n - \zeta; \mathcal{C}_1^\infty \mathcal{L}^{d,k}] + \mathcal{N}[\zeta^n - \zeta; \mathcal{C}_1^\kappa \mathcal{L}^{d,k}] + \mathcal{N}[r^n - r; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)] = 0.$$

Eventually, let $\varphi : \mathbb{R}^k \to \mathcal{L}^{d,m}$ be a C_b^2-function. Then

$$\lim_{n\to\infty} \mathcal{N}\left[\mathcal{J}(\varphi(z^n) dx^n) - \mathcal{J}(\varphi(z) dx);\, \mathcal{C}_2^\kappa(\mathbb{R}^m)\right] = 0.$$

3.4 Stochastic calculus with respect to a rough path

In this section, we will apply the previous considerations to two of the usual main aims in the theory of stochastic calculus: to study differential equations driven by a rough signal and to establish a change of variable formula.

3.4.1 Rough differential equations

Recall that we wish to solve equations of the form (28). In our algebraic setting, we will rephrase this as follows: we will say that y is a solution to (28), if $y_0 = a$, $y \in \mathcal{Q}_{\kappa,a}(\mathbb{R}^l)$ and for any $0 \le s \le t \le T$ we have

$$(\delta y)_{st} = \mathcal{J}_{st}(\sigma(y)\, dx), \tag{45}$$

where the integral $\mathcal{J}(\sigma(y)\, dx)$ has to be understood in the sense of Proposition 3.10. Our existence and uniqueness result reads as follows:

Theorem 3.13. *Let x be a process satisfying Hypothesis 3.7 and $\sigma : \mathbb{R}^l \to \mathcal{L}^{d,l}$ be a C^2 function, which is bounded together with its derivatives. Then*

1. *Equation (45) admits a unique solution y in $\mathcal{Q}_{\kappa,a}(\mathbb{R}^l)$ for any $\kappa < \gamma$ such that $2\kappa + \gamma > 1$.*
2. *The mapping $(a, x, \mathbf{x}^{\mathbf{2}}) \mapsto y$ is continuous from $\mathbb{R}^l \times \mathcal{C}_1^\gamma(\mathbb{R}^d) \times \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ to $\mathcal{Q}_{\kappa,a}(\mathbb{R}^l)$.*

Proof. We will identify the solution on a small interval $[0,\tau]$ as the fixed point of the map $\Gamma : \mathcal{Q}_{\kappa,a}(\mathbb{R}^l) \to \mathcal{Q}_{\kappa,a}(\mathbb{R}^l)$ defined by $\Gamma(z) = \hat{z}$ with $\hat{z} = a$ and $\delta\hat{z} = \mathcal{J}(\sigma(z)\, dx)$. The first step in this direction is to show that the ball

$$B_M = \left\{z;\, z_0 = a,\, \mathcal{N}[z; \mathcal{Q}_{\kappa,a}(\mathbb{R}^l)] \le M\right\} \tag{46}$$

is invariant under Γ if τ is small enough and M is large enough. However, due to Propositions 3.9 and 3.10 and assuming $\tau \le 1$ we have

$$\mathcal{N}[\Gamma(z); \mathcal{Q}_{\kappa,a}] \le c_{\sigma,x} \left(1 + \tau^{\gamma-\kappa} \mathcal{N}^2[z; \mathcal{Q}_{\kappa,a}]\right). \tag{47}$$

Since the set $\mathcal{A} = \{u \in \mathbb{R}_+^* : \; c_{\sigma,x}(1 + \tau^{\gamma-\kappa} u^2) \le u\}$ is not empty as soon as τ is small enough (see also the third point of the proof of Lemma 5.3 below), it is easily shown that the ball B_M defined at (46) is left invariant by Γ for τ small enough and M in $\mathcal{A}$.

Now, since we are working in B_M, the fixed point argument for Γ is a standard argument and is left to the reader. This leads to a unique solution to equation (45) on a small interval $[0, \tau]$. One is then able to obtain the unique solution on an arbitrary interval $[0, k\tau]$ with $k \ge 1$ by patching solutions on $[j\tau, (j+1)\tau]$. Notice here that an important point, which allows us to use a constant step τ, is the fact that the estimate (47) does not depend on the initial condition a, due to the fact that σ is bounded together with its derivatives.

$\square$

Remark 3.14. The case of an equation of the form

$$dy_t = b(y_t)\, dt + \sigma(y_t)\, dx_t, \qquad y_0 = a,$$

which can be written equivalently

$$\delta y = \mathcal{J}(b(y)\, dh) + \mathcal{J}(\sigma(y)\, dx), \qquad y_0 = a, \tag{48}$$

where $b : \mathbb{R}^k \to \mathbb{R}^k$ is a C_b^1-function and $h : [0, T] \to \mathbb{R}$ is defined by $h_t = t$, can be solved easily thanks to the previous results by taking one of the following two observations into account:

1. One can define the integral $\mathcal{J}(b(y)\, dh)$ in the usual Riemann sense for a weakly controlled path y. Then the fixed point argument of Theorem 3.13 can be extended trivially to the case of an equation with drift.
2. One can also define a path $\tilde{x} = (x, h^{(k)})$, where $h^{(k)}$ represents k copies of the function h, and write equation (48) as $\delta y = \mathcal{J}(\tilde{\sigma}(y)\, d\tilde{x})$, for a new matrix-valued function $\tilde{\sigma}$. The existence and uniqueness result follows then directly from Theorem 3.13 if b is a C_b^2-function.

Remark 3.15. We have stressed here the fact that one could deal in a rather elementary way with processes having a regularity $\gamma > 1/3$. However, if $\gamma > 1/2$, our algebraic setting also applies and the results we have obtained so far can be expressed in a simpler way:

Let x be a $\mathbb{R}^d$-valued γ-Hölder function with $\gamma > 1/2$, and m a function in $\mathcal{C}_1^k \mathcal{L}^{d,k}$, with $\gamma + \kappa > 1$. Define z by $z_0 = a \in \mathbb{R}^d$ and

$$\delta z = m\, \delta x - \Lambda(\delta m\, \delta x), \tag{49}$$

and set

$$\mathcal{J}(m\, dx) = \delta z. \tag{50}$$

Then:

1. z is well-defined as an element of $\mathcal{C}_2^\gamma(\mathbb{R}^k)$, and it holds:

$$|\mathcal{J}_{st}(m\,dx)| \le \|m\|_\infty \|x\|_\gamma |t-s|^\gamma + c_{\gamma,\kappa}\|m\|_\kappa \|x\|_\gamma |t-s|^{\gamma+\kappa}. \qquad (51)$$

2. The integral $\mathcal{J}(mdx)$ coincides with the usual Young integral, and in particular, it holds

$$\mathcal{J}_{st}(m\,dx) = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n} m_{t_i} (\delta x)_{t_i, t_{i+1}} \qquad (52)$$

for any $0 \le s < t \le T$, where the limit is taken over all partitions $\Pi_{st} = \{s = t_0, \ldots, t_n = t\}$ of $[s,t]$, as the mesh of the partition goes to zero.

3. Equation (45) can be solved in the Young sense whenever σ is a C_b^2-function, and the solution y satisfies

$$\|y\|_\lambda \le c_{a,\sigma,T}\|x\|_\gamma, \qquad (53)$$

for all $\lambda < \gamma$.

3.4.2 Itô's type formula

In the sequel of this paper, it will also be essential to have a change of variable formula for weakly controlled process. This will be achieved under the following additional assumption on $\mathbf{x^2}$, which will be shown to be valid in the fractional Brownian motion case:

Hypothesis 3.16. *Let $\mathbf{x^2}$ be the area process defined in Hypothesis 3.7 and denote by $\mathbf{x^{2,s}}$ the symmetric part of $\mathbf{x^2}$, i.e. $\mathbf{x^{2,s}} = \frac{1}{2}(\mathbf{x^2} + (\mathbf{x^2})^*)$. Then we assume that for $0 \le s < t \le T$ we have*

$$\mathbf{x}^{\mathbf{2,s}}_{st} = \frac{1}{2}[\delta x]_{st} \otimes [\delta x]_{st}.$$

Remark 3.17. It is worth noticing at this point that this assumption does not involve any limit type property of the form (44) for $\mathbf{x^2}$. This will simplify the verification of the different hypothesis for the fractional Brownian motion with respect to [6, 8].

With these assumptions in mind, our change of variable formula reads as follows:

Proposition 3.18. *Assume that x satisfies Hypothesis 3.7 and 3.16. Let $m \in \mathcal{Q}_{\kappa,b}(\mathcal{L}^{d,k})$ be a process of the form (34) and let $z \in \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ be defined by $z_0 = a$ and $\delta z = \mathcal{J}(m\,dx)$, which is given by (43). Let also $f \in C_b^2(\mathbb{R}^k;\mathbb{R})$. Then $f(z_0) = f(a)$ and*

$$[\delta(f(z))]_{st} = \mathcal{J}_{st}\left((\nabla^* f(z) m)\,dx\right). \qquad (54)$$

Proof. The strategy of our proof is quite straightforward. By using the composition and integration rules for weakly controlled processes we will compute the decompositions of $\delta(f(z))$ and $\mathcal{J}((\nabla^* f(z)m)\,dx)$ respectively, and then show that they coincide. Let us begin with the decomposition of $\delta(f(z))$. Recall that the decomposition (42) of δm can be written as

$$\delta m^i = \left[\mu^i\,\delta x\right]^* + r^i, \qquad i = 1, \ldots, k,$$

and that we have $\delta z^i = \mathcal{J}(m^i\,dx)$. Thus, for $i \le k$ the decomposition of z^i is given by

$$\delta z^i = m^i\,\delta x + \mu^i \cdot \mathbf{x^2} - \Lambda\left(r^i\,\delta x + \delta\mu^i \cdot \mathbf{x^2}\right).$$

In the sequel of the proof, we will use the following notation: we write $\hat r$ for any increment in $\mathcal{C}_2^{\mu}$ with $\mu \ge 2\kappa + \gamma > 1$, whose exact expression can change from line to line. In the same spirit, we will denote by $\hat r^{2\kappa}$ any increment in $\mathcal{C}_2^{2\kappa}$ with regularity at least 2κ. With these conventions in mind, some elementary algebraic manipulations yield

$$\begin{aligned}
&[\delta f(z)]_{st} \\
&= \sum_{i=1}^{k} \partial_i f(z_s)\left[\delta z^i\right]_{st} + \frac{1}{2}\sum_{i,j=1}^{k} \partial^2_{ij} f(z_s)\left[\delta z^i\right]_{st}\left[\delta z^j\right]_{st} + \hat r_{st} \\
&= \sum_{i=1}^{k} \partial_i f(z_s)\left(m^i_s\,(\delta x)_{st} + \mu^i_s \cdot \mathbf{x^2_{st}}\right) + \frac{1}{2}\sum_{i,j=1}^{k} \partial^2_{ij} f(z_s)\left[m^i_s(\delta x)_{st} m^j_s(\delta x)_{st}\right] + \hat r_{st} \qquad (55)\\
&= \sum_{i=1}^{k} \partial_i f(z_s)\left(m^i_s\,(\delta x)_{st} + \mu^i_s \cdot \mathbf{x^2_{st}}\right) + \frac{1}{2}\sum_{i,j=1}^{k} \partial^2_{ij} f(z_s)\left[(m^i_s)^* m^j_s\right] \cdot \left[(\delta x)_{st} \otimes (\delta x)_{st}\right] + \hat r_{st}.
\end{aligned}$$

which is the decomposition we were looking for $\delta f(z)$.

Let us compute now the decomposition of $\delta[\nabla^* f(z) m]$. We have due to Proposition 3.5 that

$$\delta[\nabla^* f(z) m]_{st} = \sum_{i=1}^{k} \delta\left[\partial_i f(z_s)\right]_{st} m^i_t + \partial_i f(z_s)\left[\delta m^i\right]_{st}. \qquad (56)$$

Recall also that, setting $m = [m^1, \ldots, m^k]^*$, δz can be decomposed into $\delta z = m\delta x + \hat r^{2\kappa}$. Thus, according to Proposition 3.9, one gets

$$\delta g(z) = \sum_{j=1}^{k} \partial_j g(z)\, m^j\,\delta x + \hat r^{2\kappa},$$

for any smooth function $g : \mathbb{R}^k \to \mathbb{R}$. Plugging this equality into (56), we obtain that

$$\delta[\nabla^* f(z) m]_{st} = A_{st} + B_{st} + \hat r^{2\kappa}_{st},$$

where

$$A_{st} = \sum_{i,j=1}^{k} \partial^k_{ij} f(z_s) m^j_s (\delta x)_{st} m^i_s, \quad \text{and} \quad B_{st} = \sum_{i=1}^{k} \partial_i f(z_s)\left[\mu^i_s (\delta x)_{st}\right]^*.$$

Notice that in the definition of A_{st}, m^i_t has been replaced by m^i_s, since the difference between the two expressions is again a remainder $\hat r^{2\kappa}$. Now, a little elementary linear algebra shows that

$$A_{st} = (\delta x)^*_{st} \sum_{i,j=1}^{k} \partial^k_{ij} f(z_s) (m^j_s)^* m^i_s,$$

and hence the decomposition of $\nabla^* f(z)m$ as a weakly controlled process can be written as

$$\delta[\nabla^* f(z)m] = [\nu\delta x]^* + \hat{r}^{2\kappa}, \tag{57}$$

with

$$\nu_s = \sum_{i,j=1}^{k} \partial_{ij}^k f(z_s)(m_s^i)^* m_s^j + \sum_{i=1}^{k} \partial_i f(z_s)\mu_s^i.$$

With the expression (57) at hand, we are now ready to compute $\mathcal{J}(\nabla^* f(z)m\, dx)$. Indeed, using Proposition 3.10 we get

$$\begin{aligned}
&\mathcal{J}_{st}(\nabla^* f(z)m\, dx) = \nabla^* f(z_s)\, m_s\, (\delta x)_{st} + \nu_s \cdot \mathbf{x}^{\mathbf{2}}_{st} + \hat{r}_{st} \\
&= \sum_{i=1}^{k} \partial_i f(z_s) \left(m_s^i\, (\delta x)_{st} + \mu_s^i \cdot \mathbf{x}^{\mathbf{2}}_{st} \right) + \sum_{i,j=1}^{k} \partial_{ij}^2 f(z_s) \left[(m_s^i)^* m_s^j \right] \cdot \mathbf{x}^{\mathbf{2}}_{st} + \hat{r}_{st}.
\end{aligned} \tag{58}$$

If we now put the expressions (55) and (58) together, we end up with

$$\begin{aligned}
&[\delta f(z)]_{st} - \mathcal{J}_{st}(\nabla^* f(z)m\, dx) \\
&\qquad = \sum_{i,j=1}^{k} \partial_{ij}^2 f(z_s) \left[(m_s^i)^* m_s^j \right] \cdot \left[\mathbf{x}^{\mathbf{2}}_{st} - \frac{1}{2} \left[(\delta x)_{st} \otimes (\delta x)_{st} \right] \right] + \hat{r}_{st}.
\end{aligned} \tag{59}$$

Let us show that this last expression depends only on the symmetric part $\mathbf{x}^{\mathbf{2},\mathbf{s}}$ of $\mathbf{x}^{\mathbf{2}}$. Indeed, it is easily checked that, if H is a symmetric matrix of $\mathbb{R}^{d,d}$, $\{M^i; i \le k\}$ a family of elements of $\mathbb{R}^{1,d}$ and $X \in \mathbb{R}^{d,d}$, then

$$\begin{aligned}
&\sum_{i,j=1}^{k} H(i,j) \left[(M^i)^* M^j \right] \cdot X \\
&\qquad = \sum_{i,j=1}^{k} \sum_{\alpha,\beta=1}^{d} H(i,j) M^i(\alpha) M^j(\beta) X(\alpha,\beta) = \sum_{i,j=1}^{k} H(i,j) \left[(M^i)^* M^j \right] \cdot X^{\mathbf{s}},
\end{aligned}$$

where $X^{\mathbf{s}}$ denotes the symmetric part of X. Applying this identity to $H = \mathrm{Hess}(f(x_s))$, $M^i = m_s^i$ and $X = \mathbf{x}^{\mathbf{2}}_{st}$, equation (59) becomes

$$\begin{aligned}
&[\delta f(z)]_{st} - \mathcal{J}_{st}(\nabla^* f(z)m\, dx) \\
&\qquad = \sum_{i,j=1}^{k} \partial_{ij}^2 f(z_s) \left[(m_s^i)^* m_s^j \right] \cdot \left[\mathbf{x}^{\mathbf{2},\mathbf{s}}_{st} - \frac{1}{2} \left[(\delta x)_{st} \otimes (\delta x)_{st} \right] \right] + \hat{r}_{st} = \hat{r}_{st},
\end{aligned}$$

thanks to Hypothesis 3.16. We thus have shown that

$$\delta f(z) - \mathcal{J}(\nabla^* f(z)m\, dx) = \hat{r}, \tag{60}$$

for an increment $\hat{r} \in \mathcal{C}_2^{\mu}$ with $\mu > 1$. Now we are in the position to prove easily that $\hat{r} = 0$. If we apply δ to the expression above, we find that $\hat{r} \in \ker \delta$. Thanks to Lemma

3.1, there exists a function $g \in \mathcal{C}_1$ such that $\hat{r} = \delta g$. Moreover, g inherits the regularity of $\hat{r}$, and hence $g \in \mathcal{C}_1^{\mu}$ with $\mu > 1$, which means that g is a constant function and that $\hat{r} = \delta g = 0$. Putting these considerations and equation (60) together, we finally get

$$\delta f(z) - \mathcal{J}(\nabla^* f(z) m\, dx) = 0,$$

which finishes our proof. $\square$

3.5 Application to the fractional Brownian motion

All the previous constructions rely on the specific assumptions we have made on the process x. In this section, we will show that the results given at Sections 3.3 and 3.4 apply to the fractional Brownian motion.

The combination of the following proposition with the results for the general theory allows us to prove Theorem 2.1.

Proposition 3.19. *Let B be a d-dimensional fractional Brownian motion and suppose $H > 1/3$. Then almost all sample paths of B satisfy the Hypothesis 3.7 and 3.16.*

Proof. Let us first check Hypothesis 3.7. It is a classical fact that $B \in \mathcal{C}_1^{\gamma}$ for any $1/3 < \gamma < H$, when B is a fractional Brownian motion with $H > 1/3$. As far as $\mathbf{x^2}$ is concerned, a natural choice is

$$\mathbf{x}^{\mathbf{2}}_{st} = \int_s^t dB_u \otimes \int_s^u dB_v, \quad \text{i. e.} \quad \mathbf{x}^{\mathbf{2}}_{st}(i,j) = \int_s^t dB_u^i \int_s^u dB_v^j, \quad i,j \in \{1,\ldots,d\},$$

for $0 \le s < t \le T$, where the stochastic integrals are understood in the Stratonovich sense. Then it is a classical result that $\mathbf{x^2}$ is well-defined for $H > 1/3$ (see, e.g., [20] for $i \neq j$ and [5] for $i = j$). The substitution formula for Stratonovich integrals also easily yields that $\delta \mathbf{x^2} = \delta x \otimes \delta x$. Furthermore, by stationarity (6) and the scaling property (5) of the fractional Brownian motion, we have that

$$\mathrm{E}\left[|\mathbf{x}^{\mathbf{2}}_{st}(i,j)|^2\right] = |t-s|^{4H} \mathrm{E}\left[|\mathbf{x}^{\mathbf{2}}_{01}(i,j)|^2\right] \le c|t-s|^{4H}.$$

¿From this inequality and thanks to the fact that $\mathbf{x^2}$ is a process in the second chaos of the fractional Brownian motion B, on which all L^p norms are equivalent for $p > 1$, we get that

$$\mathrm{E}\left[|\mathbf{x}^{\mathbf{2}}_{st}(i,j)|^p\right] \le c_p |t-s|^{2pH}. \tag{61}$$

In order to conclude that $\mathbf{x^2} \in \mathcal{C}_2^{2\gamma}\mathcal{L}^{d,d}$ for any $\gamma < 1/3$, let us recall the following inequality from [10]: let $g \in \mathcal{C}_2(V)$ for a given Banach space V; then, for any $\kappa > 0$ and $p \ge 1$ we have

$$\|g\|_\kappa \le c\left(U_{\kappa+2/p;p}(g) + \|\delta g\|_\gamma\right) \quad \text{with} \quad U_{\gamma;p}(g) = \left(\int_0^T \int_0^T \frac{|g_{st}|^p}{|t-s|^{\gamma p}}\right)^p. \tag{62}$$

By plugging inequality (61) into (62) and recalling that $\delta \mathbf{x^2} = \delta x \otimes \delta x$, we obtain that $\mathbf{x^2} \in \mathcal{C}_2^{2\gamma}\mathcal{L}^{d,d}$ for any $\gamma < H$, which shows that B satisfies Hypothesis 3.7.

The proof of Hypothesis 3.16 is now a consequence of the Itô-Stratonovich formula for the fractional Brownian motion (see, e.g., [1]).

$\square$

Remark 3.20. Proposition 3.12 implies that the theory of rough paths presented here and the classical one of Lyons' type coincide for the fractional Brownian motion with $H > 1/3$. In particular, consider the multiple integrals

$$\int_s^t dB_{u_1}^{\alpha_1} \int_s^{u_1} dB_{u_2}^{\alpha_2} \cdots \int_s^{\alpha_n} dB_{u_{n-1}}^{\alpha_{n-1}}, \quad \text{for} \quad (\alpha_1, \dots, \alpha_n) \in \{0, \dots, d\}^n,$$

with the convention $B_t^0 = t$. Then these multiple integrals, which are constructed by means of Proposition 3.10, coincide with the usual Stratonovich integral with respect to the fractional Brownian motion, see, e.g., [1, 2].

4 Malliavin calculus with respect to fBm

In this section, we assume that the Hurst index of B verifies $H > 1/2$. Let us give a few facts about the Gaussian structure of fractional Brownian motion and its Malliavin derivative process, following Chapter 1.2 in [17] and Section 2 in [19]. Let $\mathcal{E}$ be the set of step-functions on $[0, T]$ with values in $\mathbb{R}^d$. Consider the Hilbert space $\mathcal{H}$ defined as the closure of $\mathcal{E}$ with respect to the scalar product induced by

$$\left\langle (\mathbf{1}_{[0,t_1]}, \dots, \mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]}, \dots, \mathbf{1}_{[0,s_d]}) \right\rangle_{\mathcal{H}} = \sum_{i=1}^d R_H(t_i, s_i), \quad s_i, t_i \in [0, T], \; i = 1, \dots, d,$$

where $R_H(t, s)$ is given by (4). The scalar product between two elements $\phi, \psi \in \mathcal{E}$ is given by

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \gamma_H \sum_{i=1}^d \int_0^1 \int_0^1 \varphi^i(r) \psi^i(u) |r - u|^{2H-2} \, dr \, du \tag{63}$$

with $\gamma_H = H(2H - 1)$. The space $\mathcal{H}$ contains $L^{\frac{1}{H}}([0, T]; \mathbb{R}^d)$ but its elements can be distributions, see, e.g., [21]. Formula (63) holds also for $\varphi, \psi \in L^{\frac{1}{H}}([0, T]; \mathbb{R}^d)$. The mapping

$$(\mathbf{1}_{[0,t_1]}, \dots, \mathbf{1}_{[0,t_d]}) \mapsto \sum_{i=1}^d B_{t_i}^i$$

can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with $B = (B^1, \dots, B^d)$. We denote this isometry by $\varphi \mapsto B(\varphi)$. Let $\mathcal{S}$ be the set of smooth cylindrical random variables of the form

$$F = f(B(\varphi_1), \dots, B(\varphi_k)), \qquad \varphi_i \in \mathcal{H}, \quad i = 1, \dots, k,$$

where $f \in C^\infty(\mathbb{R}^k, \mathbb{R})$ is bounded with bounded derivatives. The derivative operator D of a smooth cylindrical random variable of the above form is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^k \frac{\partial f}{\partial x_i} (B(\varphi_1), \dots, B(\varphi_n)) \varphi_i.$$

This operator is closable from $L^p(\Omega)$ into $L^p(\Omega;\mathcal{H})$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 \;=\; \mathrm{E}|F|^2 + \mathrm{E}\|DF\|_{\mathcal{H}}^2.$$

In particular, if D^iF designates the Malliavin derivative of a functional $F \in \mathbb{D}^{1,2}$ with respect to B^i, we have $D^iB_t^j = \delta_{i,j}\mathbf{1}_{[0,t]}$ for $i,j = 1,\dots,d$.

The divergence operator δ is the adjoint of the derivative operator. If a random variable $u \in L^2(\Omega;\mathcal{H})$ belongs to $\mathrm{dom}(\delta)$, the domain of the divergence operator, then $\delta(u)$ is defined by the duality relationship

$$\mathrm{E}(F\delta(u)) = \mathrm{E}\langle DF, u\rangle_{\mathcal{H}}, \tag{64}$$

for every $F \in \mathbb{D}^{1,2}$. Moreover, if $u \in \mathrm{dom}(\delta)$ and $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega;\mathcal{H})$, then we have the following integration by parts formula

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathcal{H}}. \tag{65}$$

The following proposition is well known. For part (a) see, e.g., [18] and for part (b) and (c), see Proposition 19 in [19] and Theorem 3.1 in [13].

Proposition 4.1. *Let b^i and $\sigma^{i,j}$, $i = 1,\dots,n$, $j = 1,\dots,d$ be twice continuously differentiable with bounded derivatives.*
(a) Then equation (1) has a unique solution $X = (X^1,\dots,X^n)$ in the Young sense in the class of all processes having α-Hölder continuous sample paths with $1-H < \alpha < H$.
(b) It holds

$$\max_{i=1,\dots,n} \mathrm{E} \sup_{0\le t\le T} |X_t^i|^p < \infty$$

for all $p \ge 1$.
(c) Moreover, we have $X_t^i \in \mathbb{D}^{1,2}(\mathcal{H})$ for all $t \in [0,T]$, $i = 1,\dots,n$. The Malliavin derivative satisfies almost surely:

$$D_s^jX_t^i = \sigma^{i,j}(X_s) + \sum_{k=1}^n \int_s^t b_{x_k}^i(X_u)D_s^jX_u^k\,du + \sum_{k=1}^n\sum_{j=1}^d \int_s^t \sigma_{x_k}^{i,j}(X_u)D_s^jX_u^k\,dB_u^j, \quad s \le t,$$
$$D_s^jX_t^i = 0, \quad s > t,$$

for $j = 1,\dots,d$, where $D_s^jX_t^i$ is the j-th component of $D_sX_t^i$. Furthermore

$$\max_{j=1,\dots,d}\max_{i=1,\dots,n}\sup_{0\le s\le t\le T} \mathrm{E}|D_s^jX_t^i|^p < \infty.$$

5 Proof of Theorem 2.4

In the present section we will prove Theorem 2.4. We separate the proof in two parts: in the first one, we will show how to use trees for the parametrization of the expansion; while in the second one we explain how to control the remainder term, which appears when we expand $P_tf(a)$ with respect to t.

5.1 Rooted trees approach

In this section, we assume that the Hurst index of the fBm B verifies $H > 1/3$. The first step in the proof of the algebraic part of Theorem 2.4 is the following result.

Theorem 5.1. *Let $(X_t^a)_{t\in[0,T]}$ be the solution of SDE (1) with initial value $X_0^a = a \in \mathbb{R}^n$. Then for $m \in \mathbb{N}_0$ and $f \in C_b^{m+1}(\mathbb{R}^n;\mathbb{R}), b, \sigma^j, \in C_b^{m+1}(\mathbb{R}^n;\mathbb{R}^n)$, $1 \le j \le d$, we get for $t \in [0,T]$ the expansion*

$$f(X_t^a) = \sum_{\substack{\boldsymbol{t}\in LTS \\ l(\boldsymbol{t})-1\le m}} \sum_{j_1,\ldots,j_{s(\boldsymbol{t})}=1}^{d} F(\boldsymbol{t})(a)\, \mathcal{I}_{\boldsymbol{t}}(1)_{0,t} + \mathcal{R}_m(0,t) \tag{66}$$

with a truncation term

$$\mathcal{R}_m(0,t) = \sum_{\substack{\boldsymbol{t}\in LTS \\ l(\boldsymbol{t})-1=m+1}} \sum_{j_1,\ldots,j_{s(\boldsymbol{t})}=1}^{d} \mathcal{I}_{\boldsymbol{t}}(F(\boldsymbol{t})(X_s^a))_{0,t} \tag{67}$$

Proof. The proof is very similar to the proof of Theorem 4.2 in [23]. Recall we defined the differential operators $\mathcal{D}^0$ and $\mathcal{D}^j$ as

$$\mathcal{D}^0 = \sum_{k=1}^{n} b^k \frac{\partial}{\partial x^k} \qquad \text{and} \qquad \mathcal{D}^j = \sum_{k=1}^{n} \sigma^{k,j} \frac{\partial}{\partial x^k} \tag{68}$$

for $j = 1, \ldots, d$. Moreover, set

$$\Delta^k([s,t]) = \{(\tau_1,\ldots,\tau_k) \in [0,T]^k : s \le \tau_1 \le \tau_2 \le \cdots \le \tau_k \le t\}$$

for $0 \le s \le t \le T$ and $k \in \mathbb{N}$.

By reapplication of the change-of-variable formula (8), which holds true for the solution X^a to our SDE, and setting $\mathcal{D}^\alpha = \mathcal{D}^{\alpha_1} \ldots \mathcal{D}^{\alpha_k}$ and $dB^\alpha(s,s_1,\ldots,s_{k-1}) = dB_s^{\alpha_1} dB_{s_1}^{\alpha_2} \ldots dB_{s_{k-1}}^{\alpha_k}$ for a multi-index $\alpha = (\alpha_1,\ldots,\alpha_k) \in \{0,1,\ldots,d\}^k$, we get that

$$f(X_t^a) = f(a) + \sum_{k=1}^{m} \sum_{\alpha\in\{0,1,\ldots,d\}^k} \mathcal{D}^\alpha f(a) \int_{\Delta^k([0,t])} dB^\alpha(s,s_1,\ldots,s_{k-1}) + \mathcal{R}_m^*(0,t) \tag{69}$$

with the truncation term

$$\mathcal{R}_m^*(0,t) = \sum_{\alpha\in\{0,1,\ldots,d\}^{m+1}} \int_0^t \int_0^{s_m} \cdots \int_0^{s_1} \mathcal{D}^\alpha f(X_s^a)\, dB_s^{\alpha_1}\, dB_{s_1}^{\alpha_2} \ldots dB_{s_m}^{\alpha_{m+1}}. \tag{70}$$

Clearly, for $m = 0$ there exists only the tree $\mathbf{t} = \gamma \in LTS$ with $l(\mathbf{t}) - 1 = 0$ and we obtain

$$\sum_{\substack{\mathbf{t}\in LTS \\ l(\mathbf{t})-1=0}} F(\mathbf{t})(a)\, \mathcal{I}_{\mathbf{t}}(1)_{0,t} = F(\gamma)(a) = f(a). \tag{71}$$

Thus, to prove (66) it is sufficient to show for every $m \in \mathbb{N}$ that

$$\sum_{\alpha \in \{0,1,\ldots,d\}^m} \mathcal{D}^\alpha f(a) \int_{\Delta^m([0,t])} dB^\alpha(s, s_1, \ldots, s_{m-1}) = \sum_{\substack{\mathbf{t} \in LTS \\ l(\mathbf{t})-1=m}} F(\mathbf{t})(a)\ \mathcal{I}_{\mathbf{t}}(1)_{0,t}\ . \tag{72}$$

The proof proceeds by induction. Step $m = 1$ is performed for a better understanding. In this case two different trees $\mathbf{t}_1 = (\tau_0^2)^1$ and $\mathbf{t}_2 = (\tau_{j_1}^2)^1$ in LTS, all of length 2, with $\rho(\mathbf{t}_1) = 1$ and $\rho(\mathbf{t}_2) = H$ have to be considered. For these two trees we have

$$\begin{aligned}
&\sum_{\alpha \in \{0,1,\ldots,d\}} \mathcal{D}^\alpha F(\gamma)(a) \int_{\Delta^1([0,t])} dB^\alpha(s) \\
&= \sum_{k=1}^n b^k(a) \frac{\partial f}{\partial x^k}(a)\, t + \sum_{j_1=1}^d \sum_{k=1}^n \sigma^{k,j_1}(a) \frac{\partial f}{\partial x^k}(a) \int_0^t dB_s^{j_1} \\
&= F(\mathbf{t}_1)(a)\ \mathcal{I}_{\mathbf{t}_1}(1)_{0,t} + \sum_{j_1=1}^d F(\mathbf{t}_2)(a)\ \mathcal{I}_{\mathbf{t}_2}(1)_{0,t} \\
&= \sum_{\substack{\mathbf{t} \in LTS \\ l(\mathbf{t})-1=1}} \sum_{j_1,\ldots,j_{s(\mathbf{t})}=1}^d F(\mathbf{t})(a)\ \mathcal{I}_{\mathbf{t}}(1)_{0,t}\ .
\end{aligned}$$

Under the assumption that equation (72) holds for $m \in \mathbb{N}_0$ we proceed to prove the case $m+1$. Therefore, writing $\alpha = (\alpha_1, \ldots, \alpha_{m+1})$ for an element of $\{0, 1, \ldots, d\}^{m+1}$, we get

$$\begin{aligned}
&\sum_{\alpha \in \{0,1,\ldots,d\}^{m+1}} \mathcal{D}^\alpha f(a) \int_{\Delta^{m+1}([0,t])} dB^\alpha(s, s_1, \ldots, s_m) \\
&= \sum_{\substack{\mathbf{t} \in LTS \\ l(\mathbf{t})-1=m}} \sum_{j_1,\ldots,j_{s(\mathbf{t})}=1}^d \sum_{\alpha_{m+1} \in \{0,1,\ldots,d\}} \mathcal{D}^{\alpha_{m+1}} F(\mathbf{t})(a) \int_0^t \mathcal{I}_{\mathbf{t}}(1)_{0,s_m}\, dB_{s_m}^{\alpha_{m+1}}\ .
\end{aligned} \tag{73}$$

Now, we apply Lemma 2.7 in [23] to $\mathcal{D}^{\alpha_{m+1}} F(\mathbf{t})(a)$. Then, it holds for any $\mathbf{u} \in LTS$ with $l(\mathbf{u}) - 1 = m$ in the case of $\alpha_{m+1} = 0$ that

$$\mathcal{D}^{\alpha_{m+1}} \sum_{j_1,\ldots,j_{s(\mathbf{u})}=1}^d F(\mathbf{u})(a) = \sum_{k=1}^n b^k(a) \frac{\partial}{\partial x^k} \sum_{j_1,\ldots,j_{s(\mathbf{u})}=1}^d F(\mathbf{u})(a) = \sum_{\mathbf{t} \in D(\mathbf{u})} \sum_{j_1,\ldots,j_{s(\mathbf{t})}=1}^d F(\mathbf{t})(a), \tag{74}$$

where $D(\mathbf{u})$ is the set of all trees $\mathbf{t} \in LTS$ with $l(\mathbf{t}) = m + 2$, $\mathbf{t}'|_{\{2,\ldots,m+1\}} = \mathbf{u}'$, $\mathbf{t}''|_{\{1,\ldots,m+1\}} = \mathbf{u}''$ and $\mathbf{t}''(m+2) = \tau_0$. Clearly $s(\mathbf{u}) = s(\mathbf{t})$ holds for all $\mathbf{t} \in D(\mathbf{u})$.

Then we proceed by considering the case of $\alpha_{m+1} \in \{1, \ldots, d\}$. Again, by applying Lemma 2.7 in [23], we get for $\mathbf{u} \in LTS$ with $l(\mathbf{u}) - 1 = m$ that

$$\begin{aligned}
\sum_{\alpha_{m+1} \in \{1,\ldots,d\}} \mathcal{D}^{\alpha_{m+1}} \sum_{j_1,\ldots,j_{s(\mathbf{u})}=1}^d F(\mathbf{u})(a) &= \sum_{j_1,\ldots,j_{s(\mathbf{u})},\alpha_{m+1}=1}^d \sum_{k=1}^n \sigma^{k,\alpha_{m+1}}(a) \frac{\partial}{\partial x^k} F(\mathbf{u})(a) \\
&= \sum_{\mathbf{t} \in S(\mathbf{u})} \sum_{j_1,\ldots,j_{s(\mathbf{t})}=1}^d F(\mathbf{t})(a),
\end{aligned} \tag{75}$$

where $S(\mathbf{u})$ denotes the set of trees $\mathbf{t} \in LTS$ with $l(\mathbf{t}) = m+2$, $\mathbf{t}'|_{\{2,\dots,m+1\}} = \mathbf{u}'$, $\mathbf{t}''|_{\{1,\dots,m+1\}} = \mathbf{u}''$ and $\mathbf{t}''(m+2) = \tau_{j_{s(\mathbf{u})+1}}$. Here we have $s(\mathbf{t}) = s(\mathbf{u})+1$ for all $\mathbf{t} \in S(\mathbf{u})$.

Combining now the results for the case of $\alpha_{m+1} = 0$ and $\alpha_{m+1} \in \{1,\dots,d\}$, the equation

$$\sum_{\alpha_{m+1} \in \{0,1,\dots,d\}} \mathcal{D}^{\alpha^{m+1}} \sum_{j_1,\dots,j_{s(\mathbf{u})}=1}^{d} F(\mathbf{u})(a) = \sum_{\mathbf{t} \in D(\mathbf{u}) \cup S(\mathbf{u})} \sum_{j_1,\dots,j_{s(\mathbf{t})}=1}^{d} F(\mathbf{t})(a) \tag{76}$$

holds for every $\mathbf{u} \in LTS$ with $l(\mathbf{u})-1 = m$. Now it is easily seen that

$$\bigcup_{\substack{\mathbf{u} \in LTS \\ l(\mathbf{u})-1=m}} D(\mathbf{u}) \cup S(\mathbf{u}) = \{\mathbf{t} \in LTS : l(\mathbf{t})-1 = m+1\}. \tag{77}$$

As a last step, we observe that

$$\int_0^t \mathcal{I}_{\mathbf{u}}(1)_{0,s_m} \, dB_{s_m}^{\alpha_{m+1}} = \mathcal{I}_{\mathbf{t}}(1)_{0,t} \tag{78}$$

holds for all $\mathbf{t} \in D(\mathbf{u})$ in the case of $\alpha_{m+1} = 0$ and for all $\mathbf{t} \in S(\mathbf{u})$ with $\mathbf{t}''(m+1) = \tau_{\alpha_{m+1}}$ for all $\alpha_{m+1} \in \{1,\dots,d\}$. By applying (76)–(78) to (73) we thus arrive at (72) with m replaced by $m+1$, which completes the proof of the first part of Theorem 5.1.

Finally, we have to prove that $\mathcal{R}_m(0,t) = \mathcal{R}_m^*(0,t)$ given in (66) and (69). From (74)–(77) it follows that for each $\alpha \in \{0,1,\dots,d\}^{m+1}$ there exists a subset $LTS(\alpha) \subset LTS$ with a fixed choice of $j_1,\dots,j_{s(\mathbf{t})} \in \{1,\dots,d\}$ for $\mathbf{t} \in LTS(\alpha)$ such that $l(\mathbf{t})-1 = m+1$, $\mathbf{t}''(1) = \gamma$, $\mathbf{t}''(i) = \tau_{\alpha_{i-1}}$ for $i = 2,\dots,m+2$ and

$$\mathcal{D}^\alpha f(a) = \sum_{\mathbf{t} \in LTS(\alpha)} F(\mathbf{t})(a) \tag{79}$$

for all $a \in \mathbb{R}^n$ and $m \in \mathbb{N}_0$. Then, the sets $LTS(\alpha)$, $\alpha \in \{0,1,\dots,d\}^{m+1}$, build a partition of $\{\mathbf{t} \in LTS : l(\mathbf{t})-1 = m+1\}$. Thus, we have $\mathcal{I}_{\mathbf{t}}(Z_s^a)_{0,t} = \mathcal{I}_{\mathbf{u}}(Z_s^a)_{0,t}$ for all $\mathbf{t}, \mathbf{u} \in LTS(\alpha)$ and any integrable process Z. Replacing now Z_s^a by $F(\mathbf{t})(X_s^a)$ yields that for all $\alpha \in \{0,1,\dots,d\}^{m+1}$, the following relation holds true:

$$\sum_{\mathbf{t} \in LTS(\alpha)} \mathcal{I}_{\mathbf{t}}(F(\mathbf{t})(X_s^a))_{0,t} = \int_0^t \int_0^{s_m} \dots \int_0^{s_1} \mathcal{D}^\alpha f(X_s^a) \, dB_s^{\alpha_1} \, dB_{s_1}^{\alpha_2} \dots dB_{s_m}^{\alpha_{m+1}} \tag{80}$$

which completes the proof. □

Invoking Theorem 5.1 we obtain the following corollary:

Corollary 5.2. *Let $(X_t^a)_{t \in I}$ be the solution of SDE (1) with initial value $X_0^a = a \in \mathbb{R}^n$. Then for $m \in \mathbb{N}_0$ and $f \in C_b^{m+1}(\mathbb{R}^n, \mathbb{R}), b, \sigma^j \in C_b^{m+1}(\mathbb{R}^n, \mathbb{R}^n)$, $1 \leq j \leq d$, we get for $t \in [0,T]$ the expansion*

$$\mathrm{P}_t f(a) = \sum_{\substack{\boldsymbol{t} \in LTS(S) \\ l(\boldsymbol{t}) \leq m+1}} \sum_{j_1,\dots,j_{s(\boldsymbol{t})}=1}^{d} F(\boldsymbol{t})(a) \; \mathrm{E}(\mathcal{I}_{\boldsymbol{t}}(1)_{0,1}) \; t^{\rho(\boldsymbol{t})} + \mathrm{E}(\mathcal{R}_m(0,t)) \; . \tag{81}$$

Proof. Apply Theorem 5.1 and take the expectation in formula (66). Using the notation of the proof of Theorem 5.1, we observe that for each $\mathbf{t} \in LTS$ there exists an $\alpha \in \{0, 1, \ldots, d\}^m$ with $m = l(\mathbf{t}) - 1$ such that $\mathbf{t} \in LTS(\alpha)$. Then, due to (78) and the scaling property (5) it follows that

$$\mathrm{E}\left(\mathcal{I}_{\mathbf{t}}(1)_{0,t}\right) = \mathrm{E}\left(\mathcal{I}_{\mathbf{t}}(1)_{0,1}\right) t^{H\,|\alpha|+m-|\alpha|} = \mathrm{E}\left(\mathcal{I}_{\mathbf{t}}(1)_{0,1}\right)\ t^{\rho(\mathbf{t})} \tag{82}$$

holds with $|\alpha| = \sum_{i=1}^{m} 1_{\{\alpha_i \neq 0\}}$ since we have $\rho(\mathbf{t}) = H\,|\alpha| + m - |\alpha|$. $\square$

The sequel of the paper is now devoted to derive the announced controls on the remainder term $\mathrm{E}(\mathcal{R}_m(0, t))$ appearing in (81), according to the value of H and the assumptions on f, b and σ. These estimates will imply easily our Theorem 2.4.

5.2 Study of the remainder term for $1/3 < H < 1/2$

We assume in this section that $1/3 < H < 1/2$ and that assumption (A) holds true. Then we will show that for fixed $m \in \mathbb{N}$, we have

$$\mathrm{E}(\mathcal{R}_m(0, t)) = O(t^{(m+1)H}) \tag{83}$$

as $t \to 0$, a fact which trivially yields (15) in Theorem 2.4. Furthermore, notice that the control (83) is a direct consequence of the following:

Lemma 5.3. *Let $g \in \mathrm{C}^2(\mathbb{R}^n)$ be bounded together with its derivatives and X be the unique solution to (7) in $\mathcal{Q}_{\kappa,a}(\mathbb{R}^n)$ with $\kappa \in (\frac{1-H}{2}, H)$. For any $\alpha_1, \ldots, \alpha_r \in \{0, \ldots, d\}$ we have*

$$\mathrm{E}\left|\int_{\Delta^r([0,t])} g(X) dB^{\alpha_r} \ldots dB^{\alpha_1}\right| = O(t^{r-|\alpha|(1-H)}),\ \ \text{as } t \to 0, \tag{84}$$

where $|\alpha| = \sum_{i=1}^{r} 1_{\{\alpha_i \neq 0\}}$.

Proof. The more difficult setting holds when $\alpha_j \neq 0$ for all $1 \leq j \leq r$, that is when $r = |\alpha|$. For this reason we will only prove the assertion in this case. Moreover, we split the proof in several steps.

Step 1: Scaling. For $j \in \{1, \ldots, r\}$ and $c > 0$ set $B_u^{\alpha_j,(c)} = c^H\, B_{u/c}^{\alpha_j}$ and let $X^{(c)}$ denote the solution of (7), where B is replaced by $B^{(c)}$. For fixed t, we have

$$\begin{aligned}
\int_{\Delta^r([0,t])} g(X) dB^{\alpha_r} \ldots dB^{\alpha_1} &= \int_0^t dB_{t_1}^{\alpha_1} \int_0^{t_1} dB_{t_2}^{\alpha_2} \ldots \int_0^{t_{r-1}} dB_{t_r}^{\alpha_r} g(X_{t_r}) \\
&= \int_0^1 dB_{t \cdot t_1}^{\alpha_1} \int_0^{t_1} dB_{t \cdot t_2}^{\alpha_2} \ldots \int_0^{t_{r-1}} dB_{t \cdot t_r}^{\alpha_r} g(X_{t \cdot t_r}) \\
&\overset{\mathcal{L}}{=} \int_0^1 dB_{t \cdot t_1}^{\alpha_1,(t)} \int_0^{t_1} dB_{t \cdot t_2}^{\alpha_2,(t)} \ldots \int_0^{t_{r-1}} dB_{t \cdot t_r}^{\alpha_r,(t)} g(X_{t \cdot t_r}^{(t)}) \\
&= t^{rH} \int_0^1 dB_{t_1}^{\alpha_1} \int_0^{t_1} dB_{t_2}^{\alpha_2} \ldots \int_0^{t_{r-1}} dB_{t_r}^{\alpha_r} g(X_{t \cdot t_r}^{(t)}).
\end{aligned}$$

Consequently, in order to obtain (84), it suffices to prove that

$$\sup_{t \in [0,T]} \mathrm{E}\left|\int_0^1 dB_{t_1}^{\alpha_1} \int_0^{t_1} dB_{t_2}^{\alpha_2} \ldots \int_0^{t_{r-1}} dB_{t_r}^{\alpha_r}\, g(X_{t \cdot t_r}^{(t)})\right| < \infty.$$

Step 2: Fix t and set

$$z_s = \int_0^s dB_{t_r}^{\alpha_r}\, g(X_{t \cdot t_r}^{(t)}), \quad s \in [0,1].$$

By (37) and (30) we have

$$\mathcal{N}(z, \mathcal{Q}_{\kappa,0}) \le c_B \big(1 + \mathcal{N}(g(X_{t \cdot}^{(t)}), \mathcal{Q}_{\kappa,g(a)})\big) \le c_B c_g \big(1 + \mathcal{N}^2(X_{t \cdot}^{(t)}, \mathcal{Q}_{\kappa,a})\big).$$

Here, $c_B > 1$ is the random constant appearing in (37) (see also (38)), whose value will not change from line to line, while c_g denotes a non-random constant depending only on g, whose value can change from one line to another. Set now

$$q_s = \int_0^s dB_{t_{r-1}}^{\alpha_{r-1}} \int_0^{t_{r-1}} dB_{t_r}^{\alpha_r} g(X_{t \cdot t_r}^{(t)}) = \int_0^s dB_{t_{r-1}}^{\alpha_{r-1}} z_{t_{r-1}}, \quad s \in [0,1].$$

Similarly, we have

$$\mathcal{N}(q, \mathcal{Q}_{\kappa,0}) \le c_B \big(1 + \mathcal{N}(z, \mathcal{Q}_{\kappa,0})\big) \le {c_B}^2 c_g \big(1 + \mathcal{N}^2(X_{t \cdot}^{(t)}, \mathcal{Q}_{\kappa,a})\big).$$

By induction, we easily deduce that

$$\mathcal{N}\left(\int_0^{\cdot} dB_{t_1}^{\alpha_1} \int_0^{t_1} dB_{t_2}^{\alpha_2} \dots \int_0^{t_{r-1}} dB_{t_r}^{\alpha_r} g(X_{t \cdot t_r}^{(t)}), \mathcal{Q}_{\kappa,0}\right) \le {c_B}^r c_g \big(1 + \mathcal{N}^2(X_{t \cdot}^{(t)}, \mathcal{Q}_{\kappa,a})\big).$$

Since we have $|z_1| \le \|z\|_\kappa \le \mathcal{N}(z, \mathcal{Q}_{\kappa,0})$ for a path z starting from 0, we deduce from the Cauchy-Schwarz inequality that (84) is in fact a consequence of showing

$$\mathrm{E}({c_B}^{2r}) < \infty \tag{85}$$

and

$$\sup_{t \in [0,T]} \mathrm{E}\left|\mathcal{N}^4(X_{t \cdot}^{(t)}, \mathcal{Q}_{\kappa,a})\right| = \sup_{t \in [0,T]} \mathrm{E}\left|\mathcal{N}^4(X_{t \cdot}, \mathcal{Q}_{\kappa,a})\right| < \infty. \tag{86}$$

Step 3: Using that B has moments of all order and (62), we easily obtain by (38) that (85) is verified. So, let us concentrate on (86), which is more difficult. We will in fact only prove that $\mathrm{E}\big|\mathcal{N}^4(X, \mathcal{Q}_{\kappa,a})\big| < +\infty$, since we can obtain the control (86) in a similar way. Recall from the proof of Theorem 3.13 that X defined on $[0,\tau]$ belongs by definition to the ball B_M given by (46), where M and τ verify

$$M \ge c_{\sigma,B} \big(1 + \tau^{\gamma-\kappa} M^2\big).$$

For fixed τ, the inequality $u \ge c_{\sigma,B}\big(1 + \tau^{\gamma-\kappa} u^2\big)$ admits solutions u iff ${c_{\sigma,B}}^{-2} - 4\tau^{\gamma-\kappa} > 0$, i.e., iff $\tau^{\gamma-\kappa} < (4\, {c_{\sigma,B}}^2)^{-1}$. In this case, the solutions are $u \in [M_-, M_+]$, whereby

$$M_\pm = \frac{\frac{1}{c_{\sigma,B}} \pm \sqrt{\frac{1}{{c_{\sigma,B}}^2} - 4\tau^{\gamma-\kappa}}}{2\tau^{\gamma-\kappa}}.$$

By choosing for instance

$$\tau^{\gamma-\kappa} = (8\, {c_{\sigma,B}}^2)^{-1} \tag{87}$$

we obtain that

$$\mathcal{N}(X_{|_{[0,\tau]}}, \mathcal{Q}_{\kappa,a}) \leq (4-2\sqrt{2})c_{\sigma,B}. \tag{88}$$

Furthermore, as explained at the end of the proof of Theorem 3.13 and due to the crucial fact that σ and its derivatives are bounded, we can in fact choose the same M for the bound of δX on $[\tau, 2\tau]$, $[2\tau, 3\tau]$, etc. Using the triangle inequality we deduce:

$$\begin{aligned}
&\mathcal{N}(X, \mathcal{Q}_{\kappa,a}) \\
&\quad \leq \mathcal{N}(X_{|_{[0,\tau]}}, \mathcal{Q}_{\kappa,a}) + \mathcal{N}(X_{|_{[\tau,2\tau]}}, \mathcal{Q}_{\kappa,X_\tau}) + \ldots + \mathcal{N}(X_{|_{[\lfloor T\tau^{-1}\rfloor\tau,T]}}, \mathcal{Q}_{\kappa,X_{\lfloor T\tau^{-1}\rfloor\tau}}) \\
&\quad \leq (\lfloor T\tau^{-1}\rfloor + 1)M.
\end{aligned}$$

In other words, we deduce $\mathcal{N}(X, \mathcal{Q}_{\kappa,a}) \leq \mathrm{cst}\, {c_{\sigma,B}}^{1+\frac{2}{\gamma-\kappa}}$, see (87) and (88). Thus it follows easily that $\mathrm{E}\big|\mathcal{N}^4(X, \mathcal{Q}_{\kappa,a})\big| < +\infty$ and the proof of Lemma 5.3 is finished. $\square$

5.3 Some properties of iterated integrals in the case $H > 1/2$

Let us say first a few words about the strategy we have adopted in order to get equation (15): the key point will be again to get an accurate bound for $E[\mathcal{R}_m(0,t)]$, and thus we use estimates based on Malliavin calculus tools and explicit computations of moments for multiple iterated integrals with respect to the fractional Brownian motion. For a proposed pathwise control on the remainder $\mathcal{R}_m(0,t)$, see., e.g. in [11, Remark 7.4].

Before we turn to the control of the remainder in the case $H > 1/2$, we will establish first some properties of iterated integrals with respect to fractional Brownian motion. To do this, we require some additional notations.

For a multi-index $\alpha \in \{0,1,\ldots,d\}^k$ with $k \in \mathbb{N}$ denote by $l(\alpha)$ the length of α, i.e., $l(\alpha) = k$. Moreover set $\mathcal{A}_k = \{0,1,\ldots,d\}^k$ for $k \in \mathbb{N}$, i.e., $\mathcal{A}_k$ is the set of all multi-indices of length k. Furthermore, define for $\alpha \in \mathcal{A}_k$ the sets

$$\mathfrak{J}_\alpha = \{j = 1,\ldots,k : \alpha_j \neq 0\}, \qquad \text{and} \qquad \mathfrak{J}_{\alpha,i} = \{j = 1,\ldots,k : \alpha_j = i\},$$

for $i = 1,\ldots,d$ and $|\alpha| = |\mathfrak{J}_\alpha|$. Finally for a multi-index $\alpha \in \mathcal{A}_k$ and $j = 1,\ldots,k$ we denote

$$\alpha_{-j} = (\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_k).$$

Recall that for $m \in \mathbb{N}$ and $0 \leq t_1 \leq t_2 \leq T$ we set

$$\Delta^m([t_1,t_2]) = \{(\tau_1,\ldots,\tau_m) \in [0,T]^m : t_1 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq t_2\}.$$

Moreover, we will use the notation

$$\int_{\Delta^k([t_1,t_2])} dB^\alpha = \int_{t_1}^{t_2} \int_{t_1}^{s_{k-1}} \cdots \int_{t_1}^{s_1} dB_s^{\alpha_1}\, dB_{s_1}^{\alpha_2} \ldots dB_{s_k}^{\alpha_k}$$

for $\alpha \in \mathcal{A}_k$.

With these notations in hand, the following proposition is shown easily, and its proof will be omitted here. Indeed, part (a) follows immediately by the symmetry of fractional Brownian motion and part (b) can be shown analogously to Theorem 11 in [2].

Proposition 5.4. *Let $k \in \mathbb{N}$ and $\alpha \in \mathcal{A}_k$.*
(a) If $|\alpha|$ is odd, then we have

$$\mathrm{E} \int_{\Delta^k([0,1])} dB^\alpha = 0.$$

(b) If $|\alpha|$ is even, then it holds

$$\mathrm{E} \int_{\Delta^k([0,1])} dB^\alpha = \frac{(\gamma_H/2)^{|\alpha|/2}}{(|\alpha|/2)!} \sum_{\mathfrak{s} \in \mathfrak{S}_{\mathfrak{J}_\alpha}} \mathcal{V}(\mathfrak{s}, \alpha)$$

with

$$\mathcal{V}(\mathfrak{s}, \alpha) = \int_{0 \le t_1 < \ldots < t_k \le 1} \prod_{l=1}^{|\alpha|/2} \delta_{\alpha_{\mathfrak{s}(2l-1)}, \alpha_{\mathfrak{s}(2l)}} |t_{\mathfrak{s}(2l)} - t_{\mathfrak{s}(2l-1)}|^{2H-2} dt_1 \ldots dt_k,$$

where $\mathfrak{S}_{\mathfrak{J}_\alpha}$ is the group of all permutations of the set $\mathfrak{J}_\alpha$, $\gamma_H = H(2H-1)$ and $\delta_{i,j}$ is Kronecker's symbol.
(c) It holds

$$\mathrm{E} \left| \int_{\Delta^k([0,1])} dB^\alpha \right|^2 = \frac{(\gamma_H/2)^{|\alpha|}}{|\alpha|!} \sum_{\mathfrak{s} \in \mathfrak{S}^{\mathfrak{J}^2_\alpha}} \mathcal{W}(\mathfrak{s}, \alpha)$$

with

$$\mathcal{W}(\mathfrak{s}, \alpha) =$$

$$\int_{0 \le t_1 < \ldots < t_k \le 1} \int_{0 \le t_{k+1} < \ldots < t_{2k} \le 1} \prod_{l=1}^{|\alpha|} \delta_{\alpha_{\mathfrak{s}(2l-1)}, \alpha_{\mathfrak{s}(2l)}} |t_{\mathfrak{s}(2l)} - t_{\mathfrak{s}(2l-1)}|^{2H-2} \, dt_{k+1} \cdots dt_{2k} dt_1 \cdots dt_k,$$

where $\mathfrak{S}^{\mathfrak{J}^2_\alpha}$ denotes the group of all permutations of the set

$$\mathfrak{J}^2_\alpha = \{j = 1, \ldots, 2k : j \in \mathfrak{J}_\alpha \text{ or } j - k \in \mathfrak{J}_\alpha\}.$$

Notice that part (a) and (b) of the above proposition yield a representation for the coefficients $\mathrm{E}(\mathcal{I}_{\mathbf{t}}(1)_{0,1})$, since

$$\mathrm{E}(\mathcal{I}_{\mathbf{t}}(1)_{0,1}) = \mathrm{E} \int_{\Delta^{l(\mathbf{t})}([0,1])} dB^\alpha$$

with $\mathbf{t}''(i) = \alpha_i \in \{0, 1, \ldots, d\}$, $i = 1, \ldots, l(\mathbf{t})$.

For further computations, we also need the following positivity result for iterated integrals of the fractional Brownian motion.

Proposition 5.5. *Let $m_i \in \mathbb{N}$ for $i = 1, \ldots, n$ with $n \in \mathbb{N}$. Moreover let $\alpha^{m_i} \in \mathcal{A}_{m_i}$ and $0 \le s_i \le t_i \le T$ for $i = 1, \ldots, n$. It holds*

$$\mathrm{E}\left[\prod_{i=1}^n \int_{\Delta^{m_i}([s_i, t_i])} dB^{\alpha^{m_i}} \right] \ge 0.$$

Proof. Let $t_k^l = k2^{-l}$, $k = 0, 1, \ldots, 2^l$. Denote by $B^{l,(\alpha_j^{m_i})}$ the piecewise linear interpolation of $B^{(\alpha_j^{m_i})}$ with step size 2^{-l}, i.e.,

$$B_t^{l,(\alpha_j^{m_i})} = B_{t_k^l}^{(\alpha_j^{m_i})} + 2^l(t - t_k^l)(B_{t_{k+1}^l}^{(\alpha_j^{m_i})} - B_{t_k^l}^{(\alpha_j^{m_i})}), \qquad t \in [t_k^l, t_{k+1}^l).$$

We have

$$\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} dB^{\alpha^{m_i}} = \lim_{l\to\infty} \prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} dB^{l,\alpha^{m_i}}$$

almost surely, due to Proposition 3.12. Since $\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} dB^{l,\alpha^{m_i}}$ belongs to a finite Wiener chaos, we also have

$$\mathrm{E} \prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} dB^{\alpha^{m_i}} = \lim_{l\to\infty} \mathrm{E} \prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} dB^{l,\alpha^{m_i}}$$

according to [4]. Note that

$$\int_{\Delta^{m_i}([s_i,t_i])} dB^{l,\alpha^{m_i}} = \int_{s_i}^{t_i} \int_{s_i}^{t_{m_i}} \cdots \int_{s_i}^{t_2} \prod_{j=1}^{m_i} Z_{t_j}^{l,(\alpha_j^{m_i})} \, dt_1 \cdots dt_{m_i-1} \, dt_{m_i},$$

where

$$Z_t^{l,(\alpha_j^{m_i})} = 2^l(B_{t_{k+1}^l}^{(\alpha_j^{m_i})} - B_{t_k^l}^{(\alpha_j^{m_i})}), \qquad t \in [t_k^l, t_{k+1}^l)$$

for $\alpha_j^{m_i} \neq 0$ and $Z_t^{l,(0)} = 1$ for $t \in [0, T]$. We thus have

$$\begin{aligned} &\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} dB^{l,\alpha^{m_i}} \\ &\quad = \int_{\Delta^{m_n}([s_n,t_n])} \cdots \int_{\Delta^{m_1}([s_1,t_1])} \prod_{i=1}^n \prod_{j=1}^{m_i} Z_{t_j^i}^{l,(\alpha_j^{m_i})} dt_1^{m_1} \ldots dt_{m_1}^{m_1} \cdots dt_1^{m_n} \ldots dt_{m_n}^{m_n}. \end{aligned}$$

But the term $\prod_{i=1}^n \prod_{j=1}^{m_i} Z_{t_j^i}^{l,(\alpha_j^{m_i})}$ is a product, which consists only of increments of the independent fractional Brownian motions $B^1, \ldots, B^d$ with Hurst parameter $H > 1/2$ and of the constant factors 1. Since it is well-known that the increments of a fractional Brownian motion of Hurst index $H > 1/2$ are positively correlated, and also that we have, for a centered Gaussian vector $(G_1, \ldots, G_{2k})$:

$$\mathrm{E}(G_1 \ldots G_{2k}) = \frac{1}{k! 2^k} \sum_{\mathfrak{s} \in \mathfrak{S}_{2k}} \prod_{\ell=1}^k \mathrm{E}(G_{\mathfrak{s}(2\ell)} G_{\mathfrak{s}(2\ell-1)}),$$

we clearly deduce that

$$\mathrm{E} \prod_{i=1}^n \prod_{j=1}^{m_i} Z_{t_j^i}^{l,(\alpha_j^{m_i})} \geq 0$$

for all $t_1^{m_1}, \ldots, t_{m_n}^{m_n} \in [0,T]$. Hence we obtain

$$\mathrm{E} \prod_{i=1}^{n} \int_{\Delta^{m_i}([s_i,t_i])} dB^{l,\alpha^{m_i}} \geq 0$$

for every $l \in \mathbb{N}$, and the assertion follows.

$\square$

Our estimate of the remainder will also require the Malliavin derivative of an iterated integral. Recall then that for a random variable $F \in \mathbb{D}^{1,2}$ we denote by $D^i F$ the i-th component of the Malliavin derivative, i.e., $DF = (D^1 F, \ldots, D^d F)$. Recall moreover that for $\alpha \in \mathcal{A}_k$, $k \in \mathbb{N}$, we have defined

$$\mathfrak{J}_\alpha = \{j = 1, \ldots, k : \alpha_j \neq 0\}, \qquad \mathfrak{J}_{\alpha,i} = \{j = 1, \ldots, k : \alpha_j = i\},$$

for $i = 1, \ldots, d$ and

$$\alpha_{-j} = (\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_k).$$

Then the stochastic derivative of a multiple integral can be computed as follows:

Proposition 5.6. *Let $m \in \mathbb{N}$ and $\alpha \in \mathcal{A}_m$. We have*

$$\begin{aligned} D_u^i \int_{\Delta^m([s,t])} dB^\alpha(t_1, \ldots, t_m) & \\ = \sum_{j \in \mathfrak{J}_{\alpha,i}} \int_{s \leq t_1 \leq \ldots \leq t_{j-1} \leq u \leq t_j \leq \ldots \leq t_{m-1} \leq t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}) & \end{aligned} \tag{89}$$

for $i = 1, \ldots, d$.

Proof. We proceed by induction over $l(\alpha)$.

(a) Assume that $l(\alpha) = 1$. For $\alpha = (0)$ the assertion clearly holds. Moreover for $\alpha = (j)$, $j = 1, \ldots, d$, we have

$$D_u^i \int_s^t dB_\tau^{(j)} = D_u^i (B_t^{(j)} - B_s^{(j)}) = \delta_{i,j} 1_{[s,t]}(u), \quad \text{for} \quad i = 1, \ldots, d,$$

which corresponds to expression (89).

(b) Now assume that (89) holds for all multi-indices of length m and all $i = 1, \ldots, d$. For $\alpha \in \mathcal{A}_{m+1}$ we have

$$D_u^i \int_{\Delta^{m+1}([s,t])} dB^\alpha = D_u^i \int_s^t Y_\tau \, dB_\tau^{(\alpha_{m+1})}$$

with

$$Y_\tau = \int_{\Delta^m([s,\tau])} dB^{\widetilde{\alpha}},$$

where $\widetilde{\alpha} = \alpha_{-(m+1)}$. If $\alpha_{m+1} = 0$, then

$$D_u^i \int_s^t Y_\tau \, d\tau = \int_s^t D_u^i Y_\tau \, d\tau.$$

Hence we obtain by the induction assumption

$$\begin{aligned} D_u^i &\int_{\Delta^{m+1}([s,t])} dB^\alpha(t_1, \ldots, t_{m+1}) \\ &= \sum_{j \in \mathfrak{J}_{\widetilde{\alpha},i}} \int_s^t \int_{s \le t_1 \le \ldots \le t_{j-1} \le u \le t_j \le \ldots \le t_{m-1} \le \tau} dB^{\widetilde{\alpha}_{-j}}(t_1, \ldots, t_{m-1}) \, d\tau \\ &= \sum_{j \in \mathfrak{J}_{\alpha,i}} \int_{s \le t_1 \le \ldots \le t_{j-1} \le u \le t_j \le \ldots \le \tau \le t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}, \tau), \end{aligned}$$

which shows the assertion in this case. Hence it remains to consider the case $\alpha_{m+1} \neq 0$. In this case, some standard arguments based on the linear interpolation of Y and Lemma 1.2.3 in [17] yield

$$D_u^i \int_s^t Y_\tau \, dB_\tau^{(\alpha_{m+1})} = \int_s^t D_u^i Y_\tau \, dB_\tau^{(\alpha_{m+1})} + Y_u \delta_{i,\alpha_{m+1}} 1_{[s,t]}(u),$$

for $i = 1, \ldots, d$. But now we obtain by the induction assumption that

$$\begin{aligned} D_u^i &\int_s^t Y_\tau \, dB_\tau^{(\alpha_{m+1})} \\ &= \int_s^t \sum_{j \in \mathfrak{J}_{\widetilde{\alpha},i}} \int_{s \le t_1 \le \ldots \le t_{j-1} \le u \le t_j \le \ldots \le t_{m-1} \le \tau} dB^{\widetilde{\alpha}_{-j}}(t_1, \ldots, t_{m-1}) \, dB_\tau^{(\alpha_{m+1})} \\ &\quad + \delta_{i,\alpha_{m+1}} 1_{[s,t]}(u) \int_{\Delta^m([s,u])} dB^{\alpha_{-(m+1)}} \\ &= \sum_{j \in \mathfrak{J}_{\alpha,i}, j \neq m+1} \int_{s \le t_1 \le \ldots \le t_{j-1} \le u \le t_j \le \ldots \le t_{m-1} \le \tau \le t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}, \tau) \\ &\quad + \delta_{i,\alpha_{m+1}} 1_{[s,t]}(u) \int_{\Delta^m([s,u])} dB^{\alpha_{-(m+1)}} \\ &= \sum_{j \in \mathfrak{J}_{\alpha,i}} \int_{s \le t_1 \le \ldots \le t_{j-1} \le u \le t_j \le \ldots \le t_{m-1} \le t_m \le t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}, t_m), \end{aligned}$$

which is our announced relation.

□

Now, we will establish an estimate for the second moment of an iterated integral, which will be the key for the control of the remainder $\mathcal{R}_m(0,t)$ in the expansion of $P_t f(a)$. Indeed, the term $(m!)^{-1/2}$ appearing in (90) will be crucial in order to get some series convergence which will entail a nice bound on $\mathcal{R}_m(0,t)$.

Proposition 5.7. *Let $m \in \mathbb{N}$ and $\alpha \in \mathcal{A}_m$. There exists a constant $K_1 > 0$, depending only on H and T, such that*

$$\left(\mathrm{E} \left| \int_{\Delta^m([s,t])} dB^\alpha \right|^2 \right)^{1/2} \leq \frac{K_1^m}{\sqrt{m!}} |t-s|^{|\alpha|H+m-|\alpha|} \tag{90}$$

for $0 \leq s \leq t \leq T$.

Proof. The proof is separated in three steps.

(i) By stationarity (6) of the fractional Brownian motion, it follows

$$\int_{\Delta^m([s,t])} dB^\alpha \overset{\mathcal{L}}{=} \int_{\Delta^m([0,t-s])} dB^\alpha.$$

Hence we obtain by the scaling property (5) of fractional Brownian motion that

$$\int_{\Delta^m([s,t])} dB^\alpha \overset{\mathcal{L}}{=} (t-s)^{H|\alpha|+m-|\alpha|} \int_{\Delta^m([0,1])} dB^\alpha.$$

Now from Proposition 5.4 (c) it is obvious that we have

$$\mathrm{E} \left| \int_{\Delta^m([0,1])} dB^\alpha \right|^2 \leq \mathrm{E} \left| \int_{\Delta^m([0,1])} dB^{\widetilde{\alpha}} \right|^2, \tag{91}$$

where $\widetilde{\alpha}$ is given by $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_m)$ with $\widetilde{\alpha}_j = 0$ if $j \in \mathfrak{J}_{\alpha,0}$ and $\widetilde{\alpha}_j = 1$ if $j \in \mathfrak{J}_\alpha$, i.e., all integrals with respect to $B^{(i)}$, $i = 2, \ldots, n$ are replaced by integrals with respect to $B^{(1)}$.

(ii) In the next step, we will replace also the integrals with respect to t by integrals with respect to $B^{(1)}$. More precisely, we will show that

$$\mathrm{E} \left| \int_{\Delta^m([0,1])} dB^{\widetilde{\alpha}} \right|^2 \leq \gamma_H^{|\alpha|-m} \mathrm{E} \left| \int_{\Delta^m([0,1])} dB^{(1,\ldots,1)} \right|^2, \tag{92}$$

with $\gamma_H = H(2H-1)$. To prove (92) assume first that there is only one integral with respect to t, i.e. $|\mathfrak{J}_{\alpha,0}| = 1$. Thus we have

$$\int_{\Delta^m([0,1])} dB^{\widetilde{\alpha}} = \int_{\Delta^{k_1}([0,1])} \int_0^s \int_{\Delta^{k_2}([0,s])} dB^{\widetilde{\alpha}_2}\, ds\, dB^{\widetilde{\alpha}_1}$$

with $k_1 + k_2 + 1 = m$ and $\widetilde{\alpha} = (\widetilde{\alpha}_2, 0, \widetilde{\alpha}_1)$. By rearranging the order of integration, which is possible, since all integrals are pathwise defined, we get

$$\int_{\Delta^{k_1}([0,1])} \int_0^s \int_{\Delta^{k_2}([0,s])} dB^{\widetilde{\alpha}_2}\, ds\, dB^{\widetilde{\alpha}_1} = \int_0^1 Y_s\, ds,$$

where we have set

$$Y_s = \int_{\Delta^{k_1}([s,1])} \int_{\Delta^{k_2}([0,s])} dB^{\widetilde{\alpha}_2}\, dB^{\widetilde{\alpha}_1}.$$

With this notation in hand, observe that we also have

$$\int_{\Delta^m([0,1])} dB^{(1,\dots,1)} = \int_0^1 Y_s dB_s^{(1)}.$$

Hence, when $|\mathfrak{J}_{\alpha,0}| = 1$, one can recast (92) into

$$\mathrm{E}\left|\int_0^1 Y_s\, ds\right|^2 \leq \gamma_H \mathrm{E}\left|\int_0^1 Y_s dB_s^{(1)}\right|^2. \tag{93}$$

We will now proceed to the estimation of the two terms in (93): first of all, we easily get

$$\mathrm{E}\left|\int_0^1 Y_s\, ds\right|^2 = \int_0^1 \int_0^1 \mathrm{E}Y_{s_1} Y_{s_2}\, ds_1\, ds_2.$$

Let us compute now $\mathrm{E}|\int_0^1 Y_s dB^{(1)}(s)|^2$: by the relation between the Young and the divergence integral for fractional Brownian motion, see, e.g. [1] or Proposition 5.2.3 in [17], we have

$$\int_0^1 Y_s dB^{(1)}(s) = \delta^{(1)}(Y 1_{[0,1]}) + \gamma_H \int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2}\, ds_1\, ds_2,$$

where we use the notation

$$\delta^{(1)}(Y 1_{[0,1]}) = \delta\left((Y 1_{[0,1]}, \dots, 0)\right).$$

Thus we obtain

$$\begin{aligned}
\mathrm{E}\left|\int_0^1 Y_s dB^{(1)}(s)\right|^2 &= \mathrm{E}\left|\delta^{(1)}(Y 1_{[0,1]})\right|^2 + \gamma_H^2 \mathrm{E}\left|\int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2}\, ds_1\, ds_2\right|^2 \\
&\quad + 2\gamma_H \mathrm{E}\delta^{(1)}(Y 1_{[0,1]}) \int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2}\, ds_1\, ds_2 \\
&\geq \mathrm{E}\left|\delta^{(1)}(Y 1_{[0,1]})\right|^2 \\
&\quad + 2\gamma_H \mathrm{E}\delta^{(1)}(Y 1_{[0,1]}) \int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2}\, ds_1\, ds_2.
\end{aligned}$$

Since clearly $\int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2}\, ds_1\, ds_2 \in \mathbb{D}^{1,2}$, we have, owing to (64), that

$$\begin{aligned}
&\mathrm{E}\left[\delta^{(1)}(Y 1_{[0,1]}) \int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2}\, ds_1\, ds_2\right] \\
&\quad = \gamma_H \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathrm{E}\left[Y_{s_1} D^1_{s_2} D^1_{s_3} Y_{s_4}\right] |s_3 - s_4|^{2H-2} |s_1 - s_2|^{2H-2}\, ds_3\, ds_4\, ds_1\, ds_2.
\end{aligned}$$

By the definition of Y_s, $s \in [0,1]$, and applying Proposition 5.6, we can decompose the product $Y_{s_1} D^1_{s_2} D^1_{s_3} Y_{s_4}$ into a sum of products of iterated integrals, and hence

$$\mathrm{E}\left[Y_{s_1} D^1_{s_2} D^1_{s_3} Y_{s_4}\right] \geq 0, \qquad s_1, s_2, s_3, s_4 \in [0,1],$$

by Proposition 5.5. Consequently we obtain

$$\mathrm{E}\left|\int_0^1 Y_s dB^{(1)}(s)\right|^2 \geq \mathrm{E}\left|\delta^{(1)}(Y1_{[0,1]})\right|^2.$$

Furthermore, invoking [1], we get

$$\mathrm{E}\left|\delta^{(1)}(Y1_{[0,1]})\right|^2 = \gamma_H \int_0^1\int_0^1 \mathrm{E}Y_{s_1}Y_{s_2}|s_1-s_2|^{2H-2}\,ds_1\,ds_2$$
$$+\gamma_H^2\int_0^1\int_0^1\int_0^1\int_0^1 \mathrm{E}D^1_{\tau_1}Y_{s_1}D^1_{s_2}Y_{\tau_2}|s_1-s_2|^{2H-2}|\tau_1-\tau_2|^{2H-2}\,ds_1\,ds_2\,d\tau_1\,d\tau_2.$$

Besides, according to Proposition 5.5, and thanks to the fact that both $Y_{s_1}Y_{s_2}$ and $D^1_{\tau_1}Y_{s_1}D^1_{s_2}Y_{\tau_2}$ are products of iterated integrals, we obtain that

$$\mathrm{E}Y_{s_1}Y_{s_2}\geq 0, \quad \text{and} \quad \mathrm{E}D^1_{\tau_1}Y_{s_1}D^1_{s_2}Y_{\tau_2}\geq 0, \qquad s_1,s_2\in[0,1].$$

Since

$$|s_1-s_2|^{2H-2}\geq 1, \qquad s_1,s_2\in[0,1],$$

we end up with

$$\mathrm{E}\left|\int_0^1 Y_s\,dB^{(1)}(s)\right|^2 \geq \gamma_H \mathrm{E}\left|\int_0^1 Y_s\,ds\right|^2,$$

which is the announced relation (93). We have thus proved that

$$\mathrm{E}\left|\int_{\Delta^{k_1}([0,1])}\int_0^s\int_{\Delta^{k_2}([0,s])} dB^{\widetilde{\alpha}_2}\,ds\,dB^{\widetilde{\alpha}_1}\right|^2$$
$$\leq \gamma_H^{-1}\mathrm{E}\left|\int_{\Delta^{k_1}([0,1])}\int_0^s\int_{\Delta^{k_2}([0,s])} dB^{\widetilde{\alpha}_2}\,dB^{(1)}\,dB^{\widetilde{\alpha}_1}\right|^2.$$

Applying this procedure $m-|\alpha|$ times to replace all integrals with respect to t, equation (92) is now easily checked.

(iii) Let us conclude our proof: combining (91) and (92) yields

$$\mathrm{E}\left|\int_{\Delta^m([0,1])} dB^{\alpha}\right|^2 \leq \frac{\gamma_H^{|\alpha|}}{\gamma_H^m}\mathrm{E}\left|\int_{\Delta^m([0,1])} dB^{(1,\ldots,1)}\right|^2.$$

But clearly

$$\int_{\Delta^m([0,1])} dB^{(1,\ldots,1)} = \frac{1}{m!}(B_1)^m$$

and thus we have

$$\mathrm{E}\left|\int_{\Delta^m([0,1])} dB^{(1,\ldots,1)}\right|^2 = \frac{(2m)!}{2^m(m!)^3}.$$

Since

$$\frac{(2m)!}{2^m(m!)^2}\leq 2^m$$

the assertion (90) follows. $\square$

Putting together Propositions 5.6 and 5.7, we also get the following estimate for the second moment of the Malliavin derivative of an iterated integral.

Proposition 5.8. *Let $m \in \mathbb{N}$ and $\alpha \in \mathcal{A}_m$. There exists a constant $K_2 > 0$, depending only on H and T, such that we have*

$$\left(\mathrm{E} \left| D_u^i \int_{\Delta^m([s,t])} dB^\alpha \right|^2 \right)^{1/2} \leq |\mathfrak{J}_{\alpha,i}| \frac{K_2^{m-1}}{\sqrt{(m-1)!}} |t-s|^{(|\alpha|-1)H+m-|\alpha|} \tag{94}$$

for $i = 1, \ldots, d$ and all $0 \leq s \leq t \leq T$.

Proof. Thanks to Proposition 5.6 we have that

$$D_u^i \int_{\Delta^m([s,t])} dB^\alpha = \sum_{j \in \mathfrak{J}_{\alpha,i}} \int_{s \leq t_1 \leq \ldots \leq t_{j-1} \leq u \leq t_j \leq \ldots \leq t_{m-1} \leq t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}).$$

Thus it follows

$$\left(\mathrm{E} \left| D_u^i \int_{\Delta^m([s,t])} dB^\alpha(t_1, \ldots, t_m) \right|^2 \right)^{1/2} \leq \sum_{j \in \mathfrak{J}_{\alpha,i}} \left(\mathrm{E} \left| \int_{s \leq t_1 \leq \ldots \leq t_{j-1} \leq u \leq t_j \leq \ldots \leq t_{m-1} \leq t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}) \right|^2 \right)^{1/2} \tag{95}$$

Furthermore, it is easily checked that

$$\int_{s \leq t_1 \leq \ldots \leq t_{j-1} \leq u \leq t_j \leq \ldots \leq t_{m-1} \leq t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}) = \int_{\Delta^{l(\alpha^{j_1})}([s,u])} dB^{\alpha^{j_1}} \times \int_{\Delta^{l(\alpha^{j_2})}([u,t])} dB^{\alpha^{j_2}},$$

with $\alpha = (\alpha^{j_1}, i, \alpha^{j_2})$. Since an iterated integral belongs to a finite chaos with respect to B, all its L^p norms are equivalent. See, e.g., Theorem 1.4.1 in [17]. Thus, we obtain from Proposition 5.7 and Hölder's inequality that

$$\left(\mathrm{E} \left| \int_{s \leq t_1 \leq \ldots \leq t_{j-1} \leq u \leq t_j \leq \ldots \leq t_{m-1} \leq t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}) \right|^2 \right)^{1/2} c_{2,4} |t-s|^{|\alpha_{-j}|H+m-1-|\alpha_{-j}|} \frac{K_1^{m-1}}{\sqrt{l(\alpha^{j_1})!}\sqrt{l(\alpha^{j_2})!}}, \tag{96}$$

with a constant $c_{2,4} > 0$. Moreover, it is readily seen that

$$\frac{1}{\sqrt{l(\alpha^{j_1})!}\sqrt{l(\alpha^{j_2})!}} \leq \frac{1}{[m/2]!},$$

and according to the fact that $(2k)!/(k!)^2 \leq 2^{2k}$, we end up with

$$\frac{1}{\sqrt{l(\alpha^{j_1})!}\sqrt{l(\alpha^{j_2})!}} \leq \frac{2^{(m-1)/2}}{\sqrt{(m-1)!}}.$$

Plugging this inequality into (96) and (95), we obtain

$$\left(\mathrm{E} \left| D_u^i \int_{s\leq t_1 \leq \ldots \leq t_{j-1} \leq u \leq t_j \leq \ldots \leq t_{m-1} \leq t} dB^{\alpha_{-j}}(t_1, \ldots, t_{m-1}) \right|^2 \right)^{1/2}$$
$$\leq c_{2,4} \frac{(\sqrt{2}K_1)^{m-1}}{\sqrt{(m-1)!}} \left|\mathfrak{J}_{\alpha,i}\right| |t-s|^{(|\alpha_{-j}|H + m - 1 - |\alpha_{-j}|)},$$

and since

$$|\alpha_{-j}|H + m - 1 - |\alpha_{-j}| = (|\alpha| - 1)H + m - |\alpha|,$$

our claim (94) follows.

□

5.4 Study of the remainder term for $H > 1/2$

To avoid notational confusion we will write in the following X_t, $t \in [0, T]$, instead of X_t^a, $t \in [0, T]$, for the solution of the SDE with $X_0 = a$. Moreover, recall that X_t^i, $t \in [0, T]$, denotes the i-th component of X. Recall also that the differential operators $\mathcal{D}^0$ and $\mathcal{D}^j$ are defined as

$$\mathcal{D}^0 = \sum_{k=1}^n b^k \frac{\partial}{\partial x^k} \qquad \text{and} \qquad \mathcal{D}^j = \sum_{k=1}^n \sigma^{k,j} \frac{\partial}{\partial x^k} \tag{97}$$

for $j = 1, \ldots, d$ and that we have set $\mathcal{D}^\alpha = \mathcal{D}^{\alpha_1} \ldots \mathcal{D}^{\alpha_k}$ for a multi-index $\alpha \in \mathcal{A}_k$.

With the help of the auxiliary results contained in the previous section, we are now able to bound $\mathrm{E}\mathcal{R}_m(0, t)$ in the following way when $H > 1/2$:

Theorem 5.9. *Let $m \in \mathbb{N}$, $H > 1/2$ and assume that assumption (A) holds. Then there exists a constant $K_3 > 0$, depending only on H, T, d and n, such that*

$$|\mathrm{E}\mathcal{R}_m(0, t)| \leq (\mathcal{U}_{m+1} + \widetilde{\mathcal{U}}_{m+1}\mathcal{Y}) \frac{K_3^m t^{H(m+1)}}{\sqrt{m!}}$$

for all $t \in [0, T]$, where

$$\mathcal{U}_m = \sup_{\alpha \in \mathcal{A}_m} \sup_{0 \leq t \leq T} \left(\mathrm{E} \left| \mathcal{D}^\alpha f(X_t) \right|^2 \right)^{1/2},$$

$$\widetilde{\mathcal{U}}_m = \max_{i=1,\ldots,n} \sup_{\alpha \in \mathcal{A}_m} \sup_{0 \leq t \leq T} \left(\mathrm{E} \left| \frac{\partial}{\partial x_i} \mathcal{D}^\alpha f(X_t) \right|^2 \right)^{1/2}$$

and

$$\mathcal{Y} = \max_{i=1,\dots,n} \max_{j=1,\dots,d} \sup_{0\le u\le s\le T} \left(\mathrm{E}\left| D_u^j X_s^i \right|^4 \right)^{1/4}.$$

Notice then that the second part of Theorem 2.4 is an immediate consequence of the above estimate.

Before we can prove Theorem 5.9, we will need the following proposition, which is a straightforward consequence of Proposition 5.2.3 in [17], Proposition 4.1 and the properties of iterated integrals of fractional Brownian motion.

Proposition 5.10. *Let $m \in \mathbb{N}$, $\alpha \in \mathcal{A}_m$, $g \in C_b^2(\mathbb{R}^n;\mathbb{R})$, and set $J_\alpha(s,t) = \int_{\Delta^m([s,t])} dB^\alpha$. Then it holds*

$$\begin{aligned}
&\mathrm{E}\left(\int_0^t g(X_s) J_\alpha(s,t)\, dB_s^j \right) \\
&\quad = \gamma_H \mathrm{E}\left(\int_0^t \int_0^s \sum_{i=1}^n g_{x_i}(X_s)\, J_\alpha(s,t)\, D_u^j X_s^i \,|s-u|^{2H-2}\, du\, ds \right) \\
&\quad\quad + \gamma_H \mathrm{E}\left(\int_0^t \int_s^t g(X_s)\, D_u^j J_\alpha(s,t)\, |s-u|^{2H-2}\, du\, ds \right)
\end{aligned}$$

for $t \in [0,T]$ and $j = 1,\dots,d$.

We are now ready to prove the main result of this section.

Proof of Theorem 5.9: Note that by the proof of Theorem 5.1 we have

$$\mathcal{R}_m(0,t) = \sum_{\alpha\in\mathcal{A}_{m+1}} \int_0^t \int_0^{t_{m+1}} \dots \int_0^{t_2} \mathcal{D}^\alpha f(X_{t_1})\, dB_{t_1}^{\alpha_1}\, dB_{t_2}^{\alpha_2} \dots dB_{t_{m+1}}^{\alpha_{m+1}}.$$

(a) We first consider a single integrand. By interchanging the order of integration, which is possible since all integrals are pathwise defined, we have

$$\begin{aligned}
&\int_0^t \int_0^{t_{m+1}} \cdots \int_0^{t_2} \mathcal{D}^\alpha f(X_{t_1})\, dB_{t_1}^{\alpha_1}\, dB_{t_2}^{\alpha_2} \cdots dB_{t_{m+1}}^{\alpha_{m+1}} \\
&\quad = \int_0^t \int_{t_1}^t \int_{t_1}^{t_{m+1}} \cdots \int_{t_1}^{t_3} dB_{t_2}^{\alpha_2} \cdots dB_{t_m}^{\alpha_m}\, dB_{t_{m+1}}^{\alpha_{m+1}}\, \mathcal{D}^\alpha f(X_{t_1})\, dB_{t_1}^{\alpha_1} \\
&\quad = \int_0^t \int_{\Delta^m([s,t])} dB^{\alpha-1}\, \mathcal{D}^\alpha f(X_s)\, dB_s^{\alpha_1}.
\end{aligned}$$

Recall that

$$\mathcal{U}_m = \sup_{\alpha\in\mathcal{A}_m} \sup_{0\le t\le T} \left(\mathrm{E}|\mathcal{D}^\alpha f(X_t)|^2 \right)^{1/2}$$

and

$$\widetilde{\mathcal{U}}_m = \sup_{i=1,\dots,n} \sup_{\alpha\in\mathcal{A}_m} \sup_{0\le t\le T} \left(\mathrm{E}\left| \frac{\partial}{\partial x_i} \mathcal{D}^\alpha f(X_t) \right|^2 \right)^{1/2}.$$

If $\alpha_1 = 0$, we clearly have

$$\begin{aligned}
&\left| \mathrm{E} \int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\, dB^\alpha(t_1, \ldots, t_{m+1}) \right| \\
&\qquad \le \mathcal{U}_{m+1} \int_0^t \left(\mathrm{E} \left| \int_{\Delta^m([s,t])} dB^{\alpha_{-1}} \right|^2 \right)^{1/2} ds \\
&\qquad \le \mathcal{U}_{m+1} K_1^m \frac{t^{Hm+1}}{\sqrt{m!}}.
\end{aligned} \tag{98}$$

If $\alpha_1 \neq 0$ we have, according to Proposition 5.10:

$$\begin{aligned}
&\left| \mathrm{E} \int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\, dB^\alpha(t_1, \ldots, t_{m+1}) \right| \\
&\quad \le \gamma_H \left| \int_0^t \int_0^s \sum_{i=1}^n \mathrm{E} \frac{\partial}{\partial x_i} \mathcal{D}^\alpha f(X_s) \int_{\Delta^m([s,t])} dB^{\alpha_{-1}}\, D_u^{\alpha_1} X_s^i \, |s-u|^{2H-2}\, du\, ds \right| \\
&\qquad + \gamma_H \left| \int_0^t \int_s^t \mathrm{E}\, \mathcal{D}^\alpha f(X_s)\; D_u^{\alpha_1} \int_{\Delta^m([s,t])} dB^{\alpha_{-1}}\, |s-u|^{2H-2}\, du\, ds \right|.
\end{aligned}$$

Thus it follows

$$\begin{aligned}
&\left| \mathrm{E} \int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\, dB^\alpha(t_1, \ldots, t_{m+1}) \right| \\
&\le \sum_{i=1}^n \widetilde{\mathcal{U}}_{m+1} \gamma_H \int_0^t \int_0^s \left(\mathrm{E} \left| D_u^{\alpha_1} X_s^i \right|^4 \right)^{1/4} \left(\mathrm{E} \left| \int_{\Delta^m([s,t])} dB^{\alpha_{-1}} \right|^4 \right)^{1/4} |s-u|^{2H-2}\, du\, ds \\
&\qquad + \mathcal{U}_{m+1} \gamma_H \int_0^t \int_s^t \left(\mathrm{E} \left| D_u^{\alpha_1} \int_{\Delta^m([s,t])} dB^{\alpha_{-1}} \right|^2 \right)^{1/2} |s-u|^{2H-2}\, du\, ds.
\end{aligned}$$

So recalling that we have set

$$\mathcal{Y} = \max_{i=1,\ldots,n} \max_{j=1,\ldots,d} \sup_{0 \le u \le s \le T} \left(\mathrm{E} \left| D_u^j X_s^i \right|^4 \right)^{1/4},$$

we get

$$\begin{aligned}
&\left| \mathrm{E} \int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\, dB^\alpha(t_1, \ldots, t_{m+1}) \right| \\
&\quad \le n \widetilde{\mathcal{U}}_{m+1} \mathcal{Y} \gamma_H \int_0^t \int_0^s \left(\mathrm{E} \left| \int_{\Delta^m([s,t])} dB^{\alpha_{-1}} \right|^4 \right)^{1/4} |s-u|^{2H-2}\, du\, ds \\
&\qquad + \mathcal{U}_{m+1} \gamma_H \int_0^t \int_s^t \left(\mathrm{E} \left| D_u^{\alpha_1} \int_{\Delta^m([s,t])} dB^{\alpha_{-1}} \right|^2 \right)^{1/2} |s-u|^{2H-2}\, du\, ds.
\end{aligned}$$

Furthermore, invoking again the equivalence of L^p norms for iterated integral and Proposition 5.7, we obtain

$$\begin{aligned}\gamma_H \int_0^t \int_0^s \left(\mathrm{E} \left| \int_{\Delta^m([s,t])} dB^{\alpha_{-1}} \right|^4 \right)^{1/4} |s-u|^{2H-2} \, du \, ds \\ \leq c_{2,4} \frac{K_1^m}{\sqrt{m!}} \gamma_H \int_0^t \int_0^s |t-s|^{Hm} \, |s-u|^{2H-2} \, du \, ds \\ \leq c_{2,4} K_1^m \frac{t^{H(m+2)}}{\sqrt{m!}}.\end{aligned}$$

By Proposition 5.8, we get similarly

$$\begin{aligned}\gamma_H \int_0^t \int_s^t \left(\mathrm{E} \left| D_u^{\alpha_1} \int_{\Delta^m([s,t])} dB^{\alpha_{-1}} \right|^2 \right)^{1/2} |s-u|^{2H-2} \, du \, ds \\ \leq c_{2,4} K_2^{m-1} \frac{t^{H(m+1)}}{\sqrt{(m-1)!}}.\end{aligned}$$

Thus, we have shown for $\alpha_1 \neq 0$ the estimate

$$\begin{aligned}\left| \mathrm{E} \int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1}) \, dB^\alpha(t_1, \ldots, t_{m+1}) \right| \\ \leq n \widetilde{\mathcal{U}}_{m+1} \mathcal{Y} c_{2,4} K_1^m \frac{t^{H(m+2)}}{\sqrt{m!}} + \mathcal{U}_{m+1} c_{2,4} K_2^{m-1} \frac{t^{H(m+1)}}{\sqrt{(m-1)!}}.\end{aligned} \tag{99}$$

(b) Now we consider the complete remainder term. We have

$$\begin{aligned}|\mathrm{E}\mathcal{R}_m(0,t)| \leq \sum_{\alpha \in \mathcal{A}_{m+1}, \alpha_1 = 0} \left| \mathrm{E} \int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1}) \, dB^\alpha(t_1, \ldots, t_{m+1}) \right| \\ + \sum_{\alpha \in \mathcal{A}_{m+1}, \alpha_1 \neq 0} \left| \mathrm{E} \int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1}) \, dB^\alpha(t_1, \ldots, t_{m+1}) \right|\end{aligned}$$

Since $|\mathcal{A}_m| = (d+1)^m$, it follows by (98) and (99), that there exists a constant $K_3 > 0$ depending only on H, T, n and d such that

$$|\mathrm{E}\mathcal{R}_m(0,t)| \leq \left(\mathcal{U}_{m+1} + \mathcal{Y} \widetilde{\mathcal{U}}_{m+1} \right) K_3^m \frac{t^{H(m+1)}}{\sqrt{m!}},$$

which completes the proof.

□

Acknowledgments. Part of this work was done during the stays of authors at our respective institutions and we would like to thank them for their support and hospitality.

References

[1] E. Alòs, O. Mazet and D. Nualart (2001): *Stochastic calculus with respect to Gaussian processes.* Ann. Probab. **29**, 766-801.

[2] F. Baudoin and L. Coutin (2006): *Operators associated with a stochastic differential equation driven by fractional Brownian motions.* To appear in Stoch. Proc. Appl.

[3] G. Ben Arous (1989): *Flot et séries de Taylor stochastiques.* Probab. Theory Relat. Fields **81**, 29-77.

[4] C. Borell (1984): *On polynomial chaos and integrability.* Probab. Math. Statist. **3**, 191-203.

[5] P. Cheridito and D. Nualart (2005): *Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H in (0,1/2).* Ann. I. H. Poincaré **41**, 1049-1081.

[6] L. Coutin and Z. Qian (2002): *Stochastic rough path analysis and fractional Brownian motion.* Probab. Theory Relat. Fields **122**, 108-140.

[7] P. E. Kloeden and E. Platen (1999) *Numerical solutions of Stochastic Differential equations.* Springer Berlin, 3rd edition.

[8] T. Lyons and Z. Qian (2002): *System control and rough paths.* Oxford University Press.

[9] T. Lyons (1998): *Differential equations driven by rough signals.* Rev. Mat. Iberoamericana **14** (2), 215-310.

[10] M. Gubinelli (2004): *Controlling rough paths.* J. Funct. Anal. **216**, 86-140.

[11] M. Gubinelli (2006): *Ramification of rough paths.* Preprint.

[12] M. Gubinelli, S. Tindel (2006): *Rough evolution equation.* In preparation.

[13] Y. Hu and D. Nualart (2006): *Differential equations driven by Hölder continuous functions of order greater than 1/2.* Preprint available online at `http://arxiv.org/pdf/math.PR/0602050`.

[14] A. Neuenkirch (2006): *Reconstruction of fractional diffusions.* In preparation.

[15] I. Nourdin and T. Simon (2006): *Correcting Newton-Côtes integrals by Lévy areas.* Preprint Évry, available online at `http://arxiv.org/pdf/math.PR/0601544`.

[16] I. Nourdin and C.A. Tudor (2006): *Some linear fractional stochastic equations.* Stochastics **78** (2), 51-65.

[17] D. Nualart (2006): *The Malliavin Calculus and Related Topics.* Springer-Verlag. 2nd edition.

[18] D. Nualart and A. Răşcanu (2002): *Differential equations driven by fractional Brownian motion.* Collect. Math. **53** (1), 55-81.

[19] D. Nualart and B. Saussereau (2006): *Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion.* Preprint Barcelona.

[20] V. Pipiras and M.S. Taqqu (2000): *Integration questions related to fractional Brownian motion.* Probab. Theory Relat. Fields **118** (2), 251-291.

[21] V. Pérez-Abreu and C. Tudor (2002): *Transfer principle for stochastic fractional integral.* Bol. Soc. Mat. Mexicana **8**, 55-71.

[22] E. Platen and W. Wagner (1982): *On a Taylor formula for a class of Itô processes.* Probab. Math. Statist. **2**, 37-51.

[23] A. Rößler (2004): *Stochastic Taylor Expansions for the Expectation of Functionals of Diffusion Processes.* Stochastic Anal. Appl. **22** (6), 1553-1576.

[24] A.A. Ruzmaikina (2000): *Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion.* J. Statist. Phys. **100** (5-6), 1049-1069.

[25] M. Zähle (1998): *Integration with respect to fractal functions and stochastic calculus I.* Probab. Theory Relat. Fields **111**, 333-374.